The Aligarh Bull. of Maths.
Volume 23, Nos. 1-2, 2004

g-HOMOMORPHISMS AND MORPHISMS BETWEEN MORITA CONTEXTS

A.N. Al Kenani, N.J. Khojah and S.K. Nauman

Department of Mathematics, Faculty of Science, King Abdul Aziz University, P.O. Box 80203
Jeddah 21589, Saudi Arabia

(Received February 15, 2004)

Abstract. Morphisms between Morita contexts with different pairs of base rings are studied by adopting a comparatively generalized concept of homomorphisms between modules over different rings.

1. Introduction

Let $K_i = (A_i, M_i, N_i, B_i, <,> A_i, <,> B_i)$ be a Morita context (MC), in which A_i and B_i are rings, M_i and N_i are (A_i, B_i) bimodules, respectively, and $<,> A_i : N_i \otimes B_i\ M_i \rightarrow A_i$ and $<,> B_i : M_i \otimes A_i\ N_i \rightarrow B_i$ are the MC *maps* such that they satisfy the two associative conditions

(i) $m' < n, m > A_i = \ < m', n > B_i m$

(ii) $< n, m > A_i\, n' = n < m, n' > B_i$

In [1, p.275], Amitsur defined a map $K = (\alpha, \beta, \mu, \nu)$, between two MCs, K_1 and K_2, where $\alpha : A_1 \rightarrow A_2$ and $\beta : B_1 \rightarrow B_2$ are ring homomorphisms and $\mu : M_1 \rightarrow M_2$ and $\nu : N_1 \rightarrow N_2$ are respective bimodule homomorphisms. In this setting, the Morita elements (the pairs $< n, m > A_i$) of A_i must map to the Morita elements of A_2 and same holds with β_1 and β_2. This situation seems to be ambiguous as, in general, the bimodule homomorphisms μ and ν do not satisfy the scalar product property. So, in the following we have constructed a morphism between two MCs by adopting a concept of homomorphisms between modules defined over different rings which is obtained by *"pullback along morphisms"* (cf. [2, p.170]). We call such maps g-homomorphisms, where "g" stands for "generalized".

g-homomorphisms along with some examples and elementary properties are introduced in Section 2 and morphisms between Morita contexts are constructed and studied in Section 3. As applications, in Section 4, we have outlined some transfer of properties in the cases of PMC and nondegeneration. In the same section, morphisms between derived and induced derived contexts are studied. In fact ring extension is extented to context extension and conversely via static modules. In the end we proved a result for purity.

Unless otherwise stated all rings considered here are associative with the multiplicative identity, ring homomorphisms are identity preserving and the modules or bimodules are unital. For any ring A by the term 'M is an A-module' or 'M'_A we mean 'M is a right A-module'.

Keywords and phrases : Homomorphism, Morita contexts.
AMS Subject Classification : 16D90, 16D20, 15A69.

2. g-Homomorphisms

2.1. Definitions and Examples

g-Homomorphisms. Let A and B be rings and $\alpha : A \to B$ a ring homomorphism. Let M and N be right A and B - modules, respectively. An additive abelian group homomorphism $\mu : M \to N$ is said to be a *right generalized homomorphism*, or in short, a *right g-homomorphism*, if for every pair, $(m, a) \in M \times A$, $\mu(ma) - \mu(m)\alpha(a)$.

In order to emphasize the presence of the ring homomorphism α, we termed the above map as a *"right α-homomorphism"* or simply an it"α-homomorphism" as long as its "left" rival is not in action.

We say that, the α-homomorphism $\mu : M \to N$ is an α-monomorphism, α-epimorphism, or α-isomorphism, according as μ is an additive abelian group monomorphism, epimorphism, or isomorphism. Note that, if $B = A$ and $\alpha = 1_A$, then the 1_A-homomorphism $\mu : M \to N$ is precisely the regular A-homomorphism. Thus, we are justified to call the map defined abvoe a "generalized homomorphism".

g-Bimodule Homomorphisms. Let A, B, A', and B' be rings and let M and M' be (B, A) and (B', A') -bimodules, respectively. If $\alpha : A \to A'$ and $\beta : B \to B'$ are two ring homomorphisms, then an additive group homomorphism $\mu : M \to M'$ is said to be a *g-bimodule homomorphism*, or a *(β, α)-homomorphism*, if for every triple, $(b, m, a) \in B \times M \times A$, $\mu(bma) = \beta(b)\mu(m)\alpha(a)$.

Example 2.1.1. Let R and S be rings. Set $A = M_n(R)$, $B = M_n(S)$, $M = R^{(n)}$ and $N = S^{(n)}$. Any ring homomorphism $f : R \to S$, induces the ring homomorphism $f_{(n)} = A \to B$ defined by $f_{(n)}([a_{ij}]) = [f(a_{ij})]$ and the additive abelian group homomorphism $\mu : M \to N$ defined by $\mu([m_i]) = [f(m_i)]$. Then

$$\mu([m_i][a_{ij}]) = \mu([m_i])f_{(n)}([a_{ij}])$$

for all $[m_i] \in M$ and $[a_{ij}] \in A$. Hence μ is an $f_{(n)}$-homomorphism. If we let M to be an (R, A)-bimodule and N an (S, B)-bimodule, then $\mu : M \to N$ is an $(f, f_{(n)})$-homomorphism.

Example 2.1.2. Let $\alpha : A \to B$ be a ring homomorphism and M be a right A-module. Considering B as a left A-module, the map $\mu : M \to M \otimes_A B$, defined canonically, is an α-homomorphism. In particular, μ is injective if α is pure.

Example 2.1.3. Let R be a commutative ring and let A_1 and A_2 be R-algebras with an R-algebra homomorphism $\alpha : A_1 \to A_2$. Also assume that M_1 and M_2 are A_1 - and A_2 - modules with the R - linear maps $\sigma_{M_1} : M_1 \otimes_R A_1 \to M_1$ and $\sigma_{M_2} : M_2 \otimes_R A_2 \to M_2$, respectively. Then an R - linear map $\mu : M_1 \to M_2$ is said to be an α - homomorphism (of modules over algebras) if the rectangle

$$\begin{array}{ccc} M_1 \otimes_R A_1 & \xrightarrow{\mu \otimes \alpha} & M_2 \otimes_R A_2 \\ \sigma_{M_1}\downarrow & & \downarrow\sigma_{M_2} \\ M_1 & \xrightarrow[\mu]{} & M_2 \end{array}$$

commutes. In other words

$$(\mu \circ \sigma_{M_1}) \sum (m_i \otimes a_i) = [\sigma_{M_2} \circ (\mu \otimes \alpha)] \sum (m_i \otimes a_i)$$

$$= \sigma_{M_2}[\sum \mu(m_i) \otimes \alpha(a_i)]$$

where the ordered pair $(m_i, a_i) \in M_1 \times A_1$.

Analogously, only by reversing the arrows, one can construct g-homomorphisms between comodules over coalgebras, etc. Examples in other areas can similarly be constructed.

2.2. Some Elementary Properties

The Rng of Endomorphisms. By the term "Rng" we mean "Ring without multiplicative identity". Let $\alpha : A \to B$ be a ring homomorphism and denote by $Hom_\alpha(M, N)$ the set of all α-homomorphisms $\mu : M \to N$. Clearly, $Hom_\alpha(M, N)$ is an additive abelian group. Next, assume that A, B, and C, are rings and M, N, and P are A, B, and C - modules, respectively. If $\alpha : A \to B$ and $\beta : B \to C$ are ring homomorphisms and if $\mu \in Hom_\alpha(M, N)$ and $\nu \in Hom_\beta(N, P)$, then the composition $\nu \circ \mu \in Hom_{\beta\circ\alpha}(M, P)$.

Now let $\alpha : A \to A$ be a ring endomorphism. An abelian group endomorphism $\mu : M_1 \to M$ is called an *−alpha-endomorphism* if μ is an α-homomorphism. We write $Hom_\alpha(M, M) = End_\alpha(M)$, the set of all α-endomorphisms. Unfortunately, $End_\alpha(M)$ is not a ring, as the composition of two α-endomorphism is an α^2-endomorphism. The composition of two α-endomorphism if and only if α is an idempotent. Moreover, since $\mu(ma) = \mu(m)\alpha(a)$, for all $m \in M$, and $a \in A$, thus if $\alpha \neq 1_R$, then μ can not be an identity homomorphism on M. Hence we conclude that

Proposition 2.2.1. If $\alpha : A \to A$ is an endomorphism of rings, then for the A-module M,

(i) $End_\alpha(M)$ is a rng if and only if $\alpha(\neq I_A)$ is an idempotent.

(ii) $End_\alpha(M)$ is a ring if and only if $\alpha = I_A$. In this case we write $End_{I_A}(M) = End_A(M)$.

Proposition 2.2.1 gives us ample examples of rngs which are not rings. The rng $End_\alpha(M)$ together with the identity endomorphism I_M, that is, $End_\alpha(M) \cap \{I_M\}$, generates a ring. Note that, this extension of a ring in a ring is similar to that of the Dorroh extension.

g-Strong Homomorphisms. Assume that $\alpha : A \to B$ is a ring homomorphism and $\mu : M_A \to N_B$ an α-homomorphism. Note that the concept of g-homomorphisms immediately arises from, "pullback along α", in which N becomes an $\alpha(A)$-module and so the image $\mu(M)$ is an $\alpha(A)$-submodule of N. In general $\mu(M)$ is not a B-submodule of N. For example, $\mathbb{Z}$ is embedded in $\mathbb{Q}$ in $Mod - \mathbb{Z}$ but not in $Mod - \mathbb{Q}$. We say that an α-homomorphism $\mu : M_A \to N_B$ is an α-strong homomorphism if $\mu(M)$ is a B - submodule of N.

Example 2.2.2. According to our above definitions, if M and N are A-module and if $\mu : M \to N$ is an A-module isomorphism, then μ is I_A - strong isomorphism. If $N \leq M$ are A-modules, then the natural epimorphism $\mu : M \to M/N$ and the natural embedding $i_N : N \to M$ are I_A - strong epimorphism and I_A - strong monomorphism, respectively. In general, the term "strong" can go along with the α-homomorphism μ, if $\alpha : A \to B$ or $\mu : M \to N$ is an epimorphism.

Following are some instances where g-homomorphisms are strong homomorphisms.

Proposition 2.2.3. Let M be a divisible right A-module and N a torsion free right B-module. If $\alpha : A \to B$ is a ring homomorphism then any α-homomorphism $\mu : M \to N$ is α - strong iff α is an epimorphism.

Proof. One direction holds trivially. Assume that μ is strong. Let $b \in B$. For any $m \in M$, if $\mu(m) = n$, then $nb \in \mu(M)$, so there exists $m' \in M$, such that $\mu(m') = nb$. As M is divisible, there is an $a \in A$, such that $m' = ma$. So

$$\mu(m') = \mu(ma) = \mu(m)\alpha(a) = nb$$

This implies $n(\alpha(a) - b) = 0$. Hence $\alpha(a) = b$.

Let $\alpha : A \to B$ be a ring homomorphisms. Call an α-homomorphism $\mu : M \to N$ indecomposible if $\mu(M)$ is an indecomposible A-submodule of N. Moreover, if $\nu : M \to N$ is an α-homomorphism such that $\mu(M) \cong \nu(M)$, then we will write $\mu \cong \nu$. Also say that μ is a direct sum of μ_i and each μ_i is a direct summand of μ and denote it by $\mu = \bigoplus_{i \in \Lambda} \mu_i$ if

$$\bigoplus_{i \in A} \mu_i(M) = \mu(M)$$

It is clear that if each component μ is α-strong then μ is also α - strong.

Krull-Schmidt theorem can be expressed in terms of α - strong homomorphisms as under. For proof we refer to [4, p.115].

Proposition 2.2.4. Let $\mu : M \to N$ be a non-zero α - strong homomorphism. If $\mu(M)$ satisfies both acc and dcc, then there exist indecomposible α - strong homomorphisms $\mu_i : M \to N$, $i = 1, \cdots, n$, such that $\mu = \mu_i \oplus \cdots \oplus \mu_n$.

Proposition 2.2.5. (Krull-Schmidt Theorem) Let $o \neq \mu : M \to N$ be α-strong and N satisfy both acc and dcc, if

$$\mu = \mu_i \oplus \cdots \oplus \mu_s = \nu \oplus \cdots \oplus \nu_i$$

in which each μ_i and ν_i is indocomposible α - strong, then $s = t$ and $\mu_i \cong \nu_{\sigma(i)}$ for some permutation $\sigma(i)(i = 1, \ldots, s)$.

Tensor Product of g-Homomorphisms. Let $\alpha : A \to A'$ be a ring homomorphism and consider the modules $M = M_A$, $N = {}_AN$, $M' = M'_{A'}$ and $N' = {}_{A'}N'$. Let $\mu : M \to M'$ be right and $\nu : N \to N'$ left α - homomorphisms. Then $\mu \bar{\otimes} \nu : M \otimes_A N \to M' \otimes_{A'} N'$ can be evaluated in the usual way by the formula

$$(\mu \bar{\otimes} \nu)[\sum(m_i \otimes n_i)] = \sum[\mu(m_i) \otimes \nu(n_i)]$$

The above map is well defined, as we can see that there is no ambiguity in the uniqueness of the images under this tensor product map. In particular, if $m \in M$, $n \in N$, and $a \in A$, then the image of the identity $ma \otimes n = m \otimes an$ can be evaluated as

$$\begin{aligned}(\mu\bar{\otimes}\nu)(ma\otimes n) &= \mu(ma)\otimes\nu(n)\\ &= \mu(m)\alpha(a)\otimes\nu(n)\\ &= \mu(m)\otimes\alpha(a)\nu(n)\\ &= \mu(m)\otimes\nu(an)\\ &= (\mu\bar{\otimes}\nu)(m\otimes an)\end{aligned}$$

In order to make the tensor product of two g-bimodule homomorphisms a g-bimodule homomorphism, we lock at the following

Proposition 2.2.6. Let A, A', B, B', C, C' be rings. Let $\alpha : A \to A'$, $\beta : B \to B'$, and $\gamma : C \to C'$ be ring homomorphisms. If $\mu : {}_BM_A \to {}_{B'}M'_{A'}$ and $\nu : {}_AN_C \to {}_{A'}N'_{C'}$ and (β, α) - and (α, γ) - homomorphisms, respectively, then the tensor product of μ and ν, denoted by $\mu\bar{\otimes}\nu : M \otimes_A N \to M' \otimes_{A'} N'$, is a (β, γ) - homomorphism given by the formula

$$(\mu\bar{\otimes}\nu)[\sum(m_i \otimes n_i)] = \sum[\mu(m_i) \otimes \nu(n_i)]$$

for all $(m_i, n_i) \in M \times N$.

Proof. Clrearly, the map $\mu\bar{\otimes}\nu$ is well defined and is an additive group homomorphism. Moreover for any $b \in B$ and $c \in C$,

$$\begin{aligned}(\mu\bar{\otimes}\nu)[b\sum(m_i\otimes n_i)c] &= \sum[\mu(bm_i)\otimes\nu(n_ic)]\\ &= \sum[\beta(b)\mu(m_i)\otimes\nu(n_i)\gamma(c)]\\ &= \beta(b)[(\mu\bar{\otimes}\nu)\sum(m_i\otimes n_i)]\gamma(c)\end{aligned}$$

Hence we conclude that $\mu\bar{\otimes}\nu$ is a (β, γ) - homorphism.

Note that, the bar on the tensor is just to remind us the change of intermediate rings from A to A'.

If μ and ν are epimorphisms, then $\mu\bar{\otimes}\nu$ is an epimorphism. If any one or both of μ and ν are monomorphisms, then $\mu\bar{\otimes}\nu$ may not be monomorphism. In that case the results from purity and flatness can smoothly be transferred. For g-strong morphisms the following holds.

Proposition 2.2.7. Let μ be left β - strong and ν right γ - strong. Then $\mu\bar{\otimes}\nu$ is a (β, γ) - strong homomorphism.

3. Morphisms between Morita contexts

3.1. Morita Context Morphisms

In short we will represent an MC by the four basic ingradients (A, M, N, B), while the rest are assumed to be presented with the MC by default.

Basic Construction. Let $K_i = (A_i, M_i, N_i, B_i)$, $i = 1, 2$, be two MCs. A four fold set of maps

$$\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K_1 \to K_2$$

is said to be *Morita context morphism* from K_1 into K_2 if the following are satisfied:

(1) $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ are ring homomorphisms,

(2) $\mu : M_1 \to M_2$ and $\nu : N_1 \to N_2$ are (β, α) - and (α, β) - homomorphisms respectively,

(3) The following diagrams commute

$$\begin{array}{ccc} N_1 \otimes_{B_1} M_1 & \xrightarrow{\langle , \rangle_{A_1}} & A_1 \\ \nu\bar{\otimes}\mu \big\downarrow & & \alpha \big\downarrow \\ N_2 \otimes_{B_2} M_2 & \xrightarrow[\langle , \rangle_{A_2}]{} & A_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} M_1 \otimes_{A_1} N_1 & \xrightarrow{\langle , \rangle_{B_1}} & B_1 \\ \mu\bar{\otimes}\nu \big\downarrow & & \beta \big\downarrow \\ M_2 \otimes_{A_2} N_2 & \xrightarrow[\langle , \rangle_{B_2}]{} & B_2 \end{array}$$

via

$$\begin{array}{ccc} \sum (n_i \otimes m_i) & \longrightarrow & \sum \langle n_i, m_i \rangle_{A_1} \\ \big\downarrow & & \big\downarrow \\ \sum [\nu(n_i) \otimes \mu(m_i)] & \longrightarrow & \sum \langle \nu(n_i), \mu m_i) \rangle_{A_2} \end{array}$$

and

$$\begin{array}{ccc} \sum (m_i \otimes n_i) & \longrightarrow & \sum \langle m_i, n_i \rangle_{B_1} \\ \big\downarrow & & \big\downarrow \\ \sum \mu(m_i) \otimes \nu(n_i) & \longrightarrow & \sum \langle \mu(m_i), \nu(n_i) \rangle_{B_2} \end{array}$$

respectively.

Note that the commutativity of above diagrams is equivalent to the following identities:

$3'(i)$ $\langle , \rangle_{A_2} \circ (\nu\bar{\otimes}\mu) = \alpha \circ \langle , \rangle_{A_1}$

$3'(ii)$ $\langle , \rangle_{B_2} \circ (\mu\bar{\otimes}\nu) = \beta \circ \langle , \rangle_{B_1}$

These two identities or the commutativity of above diagrams assure that the Morita elements of A_1 (respt. of B_1) will map to the Morita elements of A_2 (respt. of B_2). Thus, $\alpha(I_1) \subseteq I_2$ and $\beta(J_1) \subseteq J_2$, where I_i and J_i are the trace ideals of the MCK_i for $i = 1, 2$.

A morphism $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ from an MCK into another MCK' is said to be an epimorphism (respt. a monomorphism) if all maps α, β, μ and ν are epimorphisms (respt. monomorphisms). In case κ is an epimorphism, we say that K' is a homomorphic image of K. If

κ is both, an epimorphism and a monomorphism, then κ_i is an isomorphism and the two contexts K and K' are isomorphic.

Associativity Under MC Morphisms. Now we demonstrate that both maps μ and ν satsify the associativity conditions of MCs. Let $m, m_1 \in M$, and $n \in N$. Then

$$\begin{aligned} \mu[\langle m,n\rangle_B m_1] &= \beta\langle m,n\rangle_B \mu(m_1) \\ &= \langle \mu(m), \nu(n)\rangle_B \mu(m_1) \\ &= \mu(m)\langle \nu(n), \mu(m_1)\rangle_{A'} \\ &= \mu(m)\alpha\langle n, m_1\rangle_A \\ &= \mu[m\langle n, m_1\rangle_A] \end{aligned}$$

In fact, in above, we have confirmed the commutativity of the diagram

$$\begin{array}{ccccc} (M \otimes_A N) \otimes_B M & \xrightarrow{\langle,\rangle_R \otimes 1_M} & B \otimes_B M & \xrightarrow{\cong} & M \\ (\mu\bar{\otimes}\nu)\bar{\otimes}\mu \downarrow & & \beta\bar{\otimes}\mu \downarrow & & \downarrow \mu \\ (M' \otimes_{A'} N') \otimes_{B'} M' & \overrightarrow{\langle,\rangle_R \otimes 1_{M'}} & B' \otimes_{B'} M' & \overrightarrow{\cong} & M' \end{array}$$

Similarly, the other symmetric of diagram is also commutative.

The Compositions of MC Morphisms. Let $K_i = (A_i, M_i, N_i, B_i)$; $i = 1, 2, 3$, be MCs and

$$\kappa_{ij} = \langle \alpha_{ij}, \mu_{ij}, \nu_{ij}, \beta_{ij}\rangle : K_i \to K_j, \ i, j = 1, 2, 3$$

MC morphisms in which $\alpha_{ij} : A_i \to A_j$ and $\beta_{ij} : B_i \to B_j$ are ring homomorphisms and $\mu_{ij} : M_i \to M_j$ and $\nu_{ij} : N_i \to N_j$ are $(\beta_{ij}, \alpha_{ij})$ and $(\alpha_{ij}, \beta_{ij})$-bimodule morphisms. The compositions of these morphisms can be obtained by chasing the following commutative diagram.

$$\begin{array}{ccc} N_1 \otimes_{B_1} M_1 & \xrightarrow{\langle,\rangle_{A_1}} & A_1 \\ \nu_{12}\bar{\otimes}\mu_{12} \downarrow & & \alpha_{12} \downarrow \\ N_2 \otimes_{B_2} M_2 & \xrightarrow{\langle,\rangle_{A_2}} & A_2 \\ \nu_{23}\bar{\otimes}\mu_{23} \downarrow & & \alpha_{23} \downarrow \\ N_3 \otimes_{B_3} M_3 & \xrightarrow{\langle,\rangle_{A_3}} & A_3 \end{array}$$

via the maps

$$\begin{array}{ccc}
\sum(n_i \otimes m_1) & \longrightarrow & \sum\langle n_i, m_i\rangle_{A_1} \\
\downarrow & & \downarrow \\
\sum[\nu_{12}(n_i) \otimes \mu_{12}(m_i)] & \longrightarrow & \sum[\nu_{12}(n_i)\mu_{12}(m_i)] \\
\downarrow & & \downarrow \\
\sum[\nu_{23}(n_i) \otimes \mu_{23}(m_i)] & \longrightarrow & \sum[\nu_{23}(n_i)\mu_{23}(m_i)]
\end{array}$$

Similarly, by interchanging the variables, the other diagram can also be considered. Hence

Proposition 3.1.1. If $\kappa_{ij} = \langle \alpha_{ij}, \mu_{ij}, \nu_{ij}, \beta_{ij}\rangle : K_i \to K_j$ are MC morphisms, then the composition $\kappa_{jk} \circ \kappa_{ij} : K_1 - rightarrow K_2$ is also an MC morphism.

Examples 3.1.2. Let $K = (A, M, N, B)$ be an MC and let M_1 and N_1 be submodules of M and N, respectively. If $K_1 = (A, M_1, N_1, B)$ is also an MC, then $\kappa = \langle 1_A, \mu, \nu, 1_B\rangle : K_1 \to K$ is a morphism of MCs K_1 into K, where μ and ν are the embeddings $\mu = i_{M_1} : M_1 \to M$ and $\nu = i_{N_1} : M_1 \to M$. In [5], Müller called K_1 a subcontext of K. If we assume $\bar{K} = (A, M/M_1, N/N_1, B)$ and $\bar{K}$ is also an MC, then $\kappa = \langle 1_A, \mu, \nu, 1_B\rangle : K \to \bar{K}$ is an MC morphism, where μ and ν are the natural epimorphisms. $\bar{K}$ is a homomorphic image of K.

Following example is a continuation of Example 2.1.1.

Example 3.1.3. Let $B_1 = R$ be any ring and $A_1 = M_n(R)$, $M_1 = R^{(n)}$ (row wise), and $N_1 = {}^{(n)}R$ (column wise). Considering M_1 a (B_1, A_1) - bimodule and N_1 an (A_1, B_1) - bimodule, one can always get an MC, $K_1 = (A_1, M_1, N_1, B_1)$ where the first MC map $\langle , \rangle_{A_1}$ is defined by the dyads

$$\langle \begin{bmatrix} n_1 \\ \vdots \\ n_n \end{bmatrix}, [m_1 \cdots m_n] \rangle_{A_1} = \begin{bmatrix} n_1 m_1 & \cdots & n_1 m_n \\ \vdots & \cdots & \vdots \\ n_n m_1 & \cdots & n_n m_n \end{bmatrix} \in A_1$$

and the second MC map $\langle , \rangle_{B_1}$ is defined by the dot product

$$\langle [m_1 \cdots m_n] , \begin{bmatrix} n_1 \\ \vdots \\ n_n \end{bmatrix} \rangle_{B_1} = m_1 n_1 + \cdots + m_n n_n \in B_1$$

If we choose another ring, say, $B_2 = S$, then on the similar pattern one can construct another MC $K_2 = (A_2, M_2, N_2, B_2)$.

Let $f : R \to S$ be a homomorphism of rings. Then

$$\kappa = \langle f_{(n)}, \mu, \nu, f\rangle$$

is a morphism of MCs from K_1 into K_2, where $f_{(n)} : A_1 \to A_2$ and $\mu : M_1 \to M_2$ are as defined in Example 2.2.1 and $\nu : N_1 \to N_2$ can similarly be defined as μ, but on column vectors. Clearly, $\kappa = \langle f_{(n)}, \mu, \nu, f\rangle$ mostly depends on $f : B_1 \to B_2$. In particular, if f is monic or epic then so is κ.

3.2. Morphisms Between Rings of Morita Contexts

For any MC $K_i = (A_i, M_i, N_i, B_i)$, let us denote its context ring by $T_i = \begin{bmatrix} A_i & N_i \\ M_i & B_i \end{bmatrix}$. Define map

$$\tau = \begin{bmatrix} \alpha & \nu \\ \mu & \beta \end{bmatrix} : T_1 \to T_2$$

by

$$\begin{bmatrix} \alpha & \nu \\ \mu & \beta \end{bmatrix} \begin{bmatrix} a & n \\ m & b \end{bmatrix} = \begin{bmatrix} \alpha(a) & \nu(n) \\ \mu(m) & \beta(b) \end{bmatrix}$$

Then we have

Examples 3.2.1. Let $K_i = (A_i, M_i, N_i, B_i)$ be MCs and $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K_1 \to K_2$ an MC morphism. Let T_i be the MC rings of K_i. Then the map $\tau = \begin{bmatrix} \alpha & \nu \\ \mu & \beta \end{bmatrix} : T_1 \to T_2$ is an identity preserving ring homomorphism. Moreover, $\text{Ker}(\tau)$ is an ideal of T_1 and if μ is (β, α) - strong and ν is (α, β) - strong, then $Im(\tau)$ is a subring of T_2. In this last case, $Im(\kappa) = (\alpha(A_1), \mu(M_1), \nu(N_1), \beta(B_1))$ is an MC and $Im(\tau)$ is the ring of the context $Im(\kappa)$.

Proof. The axiom under addition is trivial, while the axiom under multiplication is proved as follows.

$$\begin{bmatrix} a & n \\ m & b \end{bmatrix} \begin{bmatrix} a' & n' \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' + \langle n, m' \rangle_{A_1} & an' + nb' \\ ma' + bm' & \langle m, n' \rangle_{B_1} + bb' \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \alpha(a)\alpha(a') + \langle \nu(n), \mu(m') \rangle_{A_2} & \alpha(a)\nu(n') + \nu(n)\beta(b') \\ \mu(m)\alpha(a') + \beta(b)\mu(m') & \langle \mu(m), \nu(n') \rangle_{B_2} + \beta(b)\beta(b') \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(a) & \nu(n) \\ \mu(m) & \beta(b) \end{bmatrix} \begin{bmatrix} \alpha(a') & \nu(n') \\ \mu(m') & \beta(b') \end{bmatrix}$$

Remaining parts can be proved by using commutative diagrams given in the construction of the MC morphisms.

4. Applications

4.1. Projective Morita Contexts (PMC). An MC K is termed as a PMC, the abbreviation for a projective Morita context, if the two Morita context maps are surjective. K is a PMC iff it satisfies Morita Theorems I and II ([3, Section 3.12]). The term PMC is used in [7] just to shrink the phrase "Morita context satisfies Morita Theorems I and II". We also say that an MC ring T is a PMC ring if its context K is a PMC.

Theorem 4.1.1. Let $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ be a context morphism between MCs $K = (A, M, N, B)$ and $K' = (A', M', N', B')$.

(i) If K' is a PMC, α and β are monomorphisms, and μ and ν are (β, α) and (α, β) - epimorphisms, respectively, then K is a PMC.

(ii) If K is a PMC and κ an epimorphism then K' is also a PMC.

Proof. (i) Let K' be a PMC, that is the two Morita context maps $\langle , \rangle_{A'}$, and $\langle , \rangle_{B'}$ are epimorphisms. Consider the commutative daigram:

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{\langle , \rangle_B} & B \\ {\scriptstyle \mu \bar{\otimes} \nu} \downarrow & & \downarrow {\scriptstyle \beta} \\ M' \otimes_{A'} N' & \xrightarrow[\langle , \rangle_{B'}]{} & B' \end{array}$$

Since μ and ν are epic, $\mu \bar{\otimes} \nu$ is epic, also β is monic and $\langle , \rangle'_B$ is both monic and epic, so $\langle , \rangle_B$ is epic. Similarly $\langle , \rangle_A$ is also epic. Hence K is a PMC.

Proof of (ii) is similar to (i).

In this theorem in (ii) in fact we have proved that the homomorphic image of a PMC is a PMC. While in (i) we have proved its partial converse. The combined result is the following

Corollary 4.1.2. Let $K = (A, M, N, B)$ and $K' = (A, M', N', B)$ be two MCs with the common base rings A and B. If $\kappa = (1_A, \mu, \nu, 1_B) : K \to K'$ is an epimorphism, then K is a PMC.

4.2. Nondegenerate Morita Context

Recall that an MC $K = (A, M, N, B)$ is nondegenerate iff it satisfies any one of the conditions of following lemma. For the proof one may refer to [5,8, & 9]. Let us also an MC ring T nondegenerate if its context K is nondegenerate.

Lemma 4.2.1. For an MC $K = (A, M, N, B)$ the following are equivalent.

(i) M_A, N_B ${}_BM$ and ${}_AN$ are faithful and the two MC maps $\langle , \rangle_A$ and $\langle , \rangle_B$ are also faithful.

(ii) M_A is faithful and $\langle N, m \rangle_A \neq 0$ whenever $0 \neq m \in M$.

(iii) All A-modules and B-modules associated are I-free and J-free.

Theorem 4.2.2. Let $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ be a homomorphism of MCs K and K' such that α and μ are monomorphisms and ν is an epimorphism. If the MC K' (respt. MC ring T) is nondegenerate, then K (respt. T) is also nondegenerate.

Proof. Assume that $M_A a = 0_M$, for some $a \in A$. Then for all $m \in M$, $ma = 0_M$. Or

$$\mu(ma) = \mu(m)\alpha(a) = 0_{M'}$$

But $M'_{A'}$ is faithful, so $\alpha(a) = 0$ and since α is a monomorphism, $a = 0_A$. Hence M_A is faithful.

Next, assume that $\langle N, m\rangle_A = 0_A$. Then

$$\alpha\langle N, m\rangle_A = \langle \nu(N), \mu(m)\rangle_{A'} = \langle N', \mu(m)\rangle_{A'} = 0_{A'}$$

which implies that $\mu(m) = 0$. But according to the hypothesis, μ is monic, $m = 0$. Hence both conditions of Lemma 4.2.1 (ii) are satisfied and which implies that K be nondegenerate.

Theorem 4.2.3. Let $\kappa = \langle \alpha, \mu, \nu, \beta\rangle : K \to K'$ be a morphism of an MC K into another MC K' such that α and μ are isomorphisms. If K (respt. T) is nondegenerate, then K' (respt. T') is also nondegenerate.

Proof. Let the MC $K = (A, M, N, B)$ be nondegenerate. Assume that in $K' = (A', M', N', B')$, $M'a' = 0_{M'}$ for some $a' \in A'$. Since $\mu(M) \subseteq M'$ and α is an epimorphism, there exists $a \in A$ such that

$$M'a' = \mu(M)\alpha(a) = \mu(Ma) = \{0_{M'}\}$$

Since μ is monic, $Ma = \{0_M\}$ and as M_A is faithful, $a = 0_A$, which implies $a' = 0_B$.

Now assume that $\langle N', n'\rangle = \{0_{A'}\}$. Since $\nu(N) \subseteq N'$ and μ is epic, then for some $m \in M$

$$\langle \nu(N), \mu(m)\rangle_{A'} = \alpha\langle N, m\rangle = \{0_{A'}\}$$

But α is monic, so $\langle N, m\rangle = \{0_A\}$ which implies that $m = 0_M$. Hence $\mu(m) = m' = 0$, and by Lemma 4.2.1 we conclude that K' is nondegenerate.

4.3. Context Existence/Ring Extensions

This section poses another example of morphisms between Morita contexts. In fact, in the following context extensions and ring extensions are mutually studied.

Let A and B be rings and as previously, $\alpha : A \to B$, a ring homomorphism such that $\alpha(I_A) = I_B$. Assume that M is an A - module and $D = \mathrm{End}_A(M)$, the ring of endomorphisms on M_A. Next we assume that $E = \mathrm{End}_B(M \otimes_A B)$, the ring of endomorphisms on $M \otimes_A B$ in $Mod - B$. Then $M \otimes_A B$ becomes an (E, B) - bimodule, and there is a ring homomorphism $\sigma : D \to E$ defined by

$$\sigma(d)(m \otimes b) = d(m) \otimes b,$$

where $b \in B,\ d \in D$ and $m \in M$. Clearly, $\sigma(I_D) = I_E$.

The Context Induced from the Derived Contexts. Now consider the dual module $M^* = \mathrm{Hom}_A(M, A)$ of M. Let $K = (A, M, M^*, D)$ be the derived context of M. Instead of putting some conditions on M, assume that $M^* \otimes_D E$ is left B-module. We will continue this assumption up to the end. Now we claim that $K' = (B, M \otimes_A B,\ M^* \otimes_D E, E)$ is a Morita context. We call it a *context induced from the derived context of M*. Indeed

$$\begin{aligned}(M^* \otimes_D E) \otimes_E (M \otimes_A B) &\cong M^* \otimes_D M \otimes_A B \\ &\longrightarrow A \otimes_A B \\ &\cong B\end{aligned}$$

where the arrow is the MC map $\langle , \rangle_A : M^* \otimes_D M \to A$ of the first MC K. Simiarly

$$\begin{aligned}(M^* \otimes_A B) \otimes_B (M^* \otimes_D E) &\cong M \otimes_A M^* \otimes_D E \\ &\longrightarrow D \otimes_D E \\ &\cong E\end{aligned}$$

The Morphism Between Derived and Induced Contexts. Assume that $\kappa = \langle \alpha, \mu, \nu, \sigma \rangle : K \to K'$, is a map in which $\alpha : A \to B$ and $\sigma : D \to E$ are as given above, $\mu : M \to M \otimes_A B$ is defined by $\mu(m) = m \otimes 1_B$ for all $m \in M$ and $\nu : M^* \to M^* \otimes_D B$ is defined by $\nu(m^*) = m^* \otimes 1_E$. Then we have

Theorem 4.3.1. If $A, B, D, E, M, M^*, \alpha, \sigma, \mu$ and ν are as given above, then $\kappa = \langle \alpha, \mu, \nu, \beta \rangle : K \to K'$ is an MC morphism.

Proof. First we verify that μ and ν are (σ, α) - and (α, σ) - homomorphisms, respectively. Indeed, for all $a \in A$, $d \in D$, $m \in M$, and $m^* \in M^*$, we can write the following relations

$$\begin{aligned}\mu(dma) &= \sigma(d)(m \otimes 1_B)\alpha(a) \\ &= d(m) \otimes \alpha(a)\end{aligned}$$

and

$$\begin{aligned}\nu(am^*d) &= \alpha(a)(m^* \otimes 1_E)\sigma(d) \\ &= \alpha(a)[m^* \otimes \sigma(d)]\end{aligned}$$

Next we establish the commutativity of the following diagrams

$$\begin{array}{ccc} M \otimes_A M^* & \xrightarrow{\langle , \rangle_D} & D \\ {\scriptstyle \mu \bar{\otimes} \nu}\downarrow & & \downarrow{\scriptstyle \sigma} \\ (M \otimes_A B) \otimes_B (M^* \otimes_D E) & \xrightarrow{\langle , \rangle_E} & E \end{array}$$

and

$$\begin{array}{ccc} M^* \otimes_D M & \xrightarrow{\langle,\rangle_A} & A \\ \nu\bar{\otimes}\mu \big\downarrow & & \big\downarrow \alpha \\ (M^* \otimes_D E) \otimes_E (M \otimes_A B) & \xrightarrow{\langle,\rangle_B} & B \end{array}$$

In the first diagram, in one direction

$$[\sigma \circ \langle,\rangle_D] \sum (m_i \otimes m_i^*) \;=\; \sigma[\sum \langle m_i, m_i^* \rangle_D \in E$$

and from the other direction we get

$$\begin{aligned} [\langle,\rangle_E \circ \mu\bar{\otimes}\nu] \sum (m_i \otimes m_i^*) &= \langle,\rangle_E \sum [\mu(m_i) \otimes \nu(m_i^*)] \\ &= \sum \langle m_i \otimes 1_B, m_i^* \otimes 1_E \rangle_E \\ &\in E \end{aligned}$$

Note that, for any $n \in M$ and $b \in B$

$$\begin{aligned} \sigma \langle m, m^* \rangle_D (n \otimes b) &= \langle m, m^* \rangle_D n \otimes b \\ &= m[m^*(n)] \otimes b \end{aligned}$$

Similarly

$$\begin{aligned} (m \otimes 1_B) \otimes (m^* \otimes 1_E) &\longrightarrow (m \otimes m^*) \otimes 1_E \\ &\longrightarrow \langle m, m^* \rangle_D \otimes 1_E \\ &\longrightarrow \langle m, m^* \rangle_D 1_E \in E \end{aligned}$$

Then, by evaluating $n \otimes b$ at the last function, we get

$$\langle m, m^* \rangle_D 1_E (n \otimes b) = m[m^*(n)] \otimes b$$

Hence we conclude

$$\langle,\rangle_E \circ \mu\bar{\otimes}\nu = \sigma \circ \langle,\rangle_D$$

For the second diagram one can similarly prove that

$$[\alpha \circ \langle,\rangle_A] = [\langle,\rangle_B \circ \nu\bar{\otimes}\mu]$$

Hence we conclude that κ is morphism between contexts.

The following is an immediate consequence of above theorem.

Corollary 4.3.2. Let T and T' be the rings of MCs K and K', respectively. Then the MC map $\kappa : K \to K'$ of above theorem induces the ring homomorphism $\tau : T \to T'$.

4.4. Static Modules

M-Static Modules. An object V of Mod $-A$ is static if it remains invariant under the composition of the adjoint functors $\mathrm{Hom}_A(M, -)$ and $- \otimes_D M$. In particular, the ring A as an A - module is M - static if $M^* \otimes_D M \cong A$ via the natural isomorphism $m^* \otimes m \to m^*(m)$ for all $m \in M$ and $m^* \in M^*$.

In case the ring A is M − static, by [6, Lemma 3.5] we have

Lemma 4.4.1. If the ring A is M − static, then

$$M^* \otimes_D E \cong (M \otimes_A B)^*$$

as E-modules via the map

$$(m^* \otimes f)\left(\sum_{i=I}^{k} m_i \otimes b_i\right) \mapsto \sum_{j=1}^{l} \langle m^*, n_j \rangle c_j$$

where $m_i, n_j \in M$, $m^* \in M^*$ and $b_i, c_j \in B$ and $f \in E$ is such that

$$f\left(\sum_{i=I}^{k} m_i \otimes b_i\right) = \sum_{j=1}^{l} n_j \otimes c_j$$

Hence we state that

Theorem 4.4.2. If the ring A is M − static, then the induced derived contex of M is isomorphic to the derived context of $M \otimes_A B$. The respective rings of contexts are also isomorphic.

Proof. It follows from Theorem 4.3.1 and Lemma 4.4.1 that there is an MC morphism from the induced derived context of M to the derived context of $M \otimes_A B$ given by

$$\kappa' = \langle \alpha', \mu', \nu', \beta' \rangle : K' \to K''$$

where

$$K'' = \{B, M \otimes_A B, (M \otimes_A B)^*, E\}$$

Clearly, α', β' and μ' are the identical maps while

$$\nu' : M^* \otimes_D E \longrightarrow (M \otimes_A B)^*$$

is an isomorphism as given in the Lemma 4.4.1. Hence $\kappa' : K' \to K''$ is an MC isomorphism. The last statement follows from Corollary 4.3.2.

Corollary 4.4.3. If the ring A is M − static, then there always is a morphism (respt. ring homomorphism) between the derived contexts (respt. rings of derived contexts) of M and of $M \otimes_A B$.

Proof. By Proposition 3.1.3, the composition of the MC morphisms

$$K \xrightarrow{\kappa} K' \xrightarrow{\kappa'} K''$$

is an MC morphism.

If the derived context of M is a PMC, then A becomes M − static. By using Theorems 3.3 and 3.4 of [6] we restate that

Corollary 4.4.4. *(a)* If K, the derived context of M, is a PMC, then K', the induced derived context of M, and the derived context K'' of $M \otimes_A B$ are also PMCs.
(b) If $\alpha : A \to B$ is a monomorphism then K is a PMC if and only if K' (or K'') is a PMC.

4.5. Purity

Let the ring homomorphism $\alpha : A \to B$ be a pure homomorphism. Then for every $M \in$ Mod $-A$, the α-homomorphism $\mu : M \otimes_A B$ is injective (Example 2.1.2).

Recently, in studying relationship between effective descent morphisms and pure homomorphisms, Mesablishvili in [4;3.2. Theorem] proved that

Theorem 4.5.1. If $\alpha : A \to B$ is a pure homomorphism of commutative rings and if for any $M \in$ Mod $-A$, $M \otimes_A B$ is f.g., flat, and f.g. flat, and f.g. projective in Mod − B, then M is f.g., flat, f.g. flat, and f.g. projective in Mod − A, respectively.

By using Corollary 4.4.4 (b), we can add one more property in the above list without involving commutativity of rings.

Corollary 4.5.2. If $\alpha : A \to B$ is a pure (or simply injective), then M is a progenerator of Mod − A if and only if $M \otimes_A B$ is a progenerator of Mod − B.

Proof. Recall that M is a progenerator of Mod − A if and only if any arbitrary MC $K = (A, M, N, C)$ is a PMC (cf. [3 & 7]). Then ${}_AN_C \cong M^*$ and $C \cong$ End $(M_A) = D$. This holds if and only if the derived context of $M, K = (A, M, M^*, D)$ is a PMC. Note that, if $\alpha : A \to B$ is a pure then it is also injective. By Corollary 4.4.4(b), K is a PMC if and only if the induced context K' of $M \otimes_A B$ is a PMC, which holds if and only if $M \otimes_A B$ is a progenerator of Mod − B.

References

[1] Amitsur, S.A. : *Rings of quotients and Morita contexts*, J. Algebra, 17 (1971) 273-298.

[2] Hungerford, T.W. : *Algebra*, Springer-Verlag, (1974).

[3] Jacobson, N. : *Basic Algebra-II*, Freeman & Co., (1980).

[4] Mesablishvili, B. : *On some properties of pure morphisms of commutative rings*, Th. & App. of Categories, 10 (2002) 180-186.

[5] Müller, B.J. : *The quotient categories of a Morita context*, J. Algebra, 28 (1974) 389-407.

[6] Nauman, S.K. : *Static modules, stable Clifford theory, and Morita similarities of rings*, J. Algebra, 143 (1991) 498-504.

[7] Nauman, S.K. : *Static modules, stable Clifford theory, and colocalization - localization*, J. Algebra, 170 (1994) 400-421.

[8] Zhengping, Z. : *Endomorphism rings of nondegenerate modules*, Proc. AMS 120 (1994) 85-88.

[9] Zhengping, Z. : *Correspondence theorems for non-degenerate modules and their endomorphism rings*, Proc. AMS 121 (1994) 25-32.