International Journal of Applied Mathematics
Volume 10 No. 1 2002, 99-123

STABILITY ANALYSIS OF COUNTER-PROPAGATING FIELDS IN A NONLINEAR SYSTEM WITH PHASE-SENSITIVE SOURCE

H.A. Batarfi
Mathematics Dept.
Faculty of Science (Women's Section)
King Abdul Aziz University
P. O. Box 41101, Jeddah 21521
Kingdom of SAUDI ARABIA
e-mail: hbatarfi@kaau.edu.sa

Abstract: A study of the interaction of two counter-propagating beams in a two-level atomic medium which is in contact with a "phase-sensitive" (or squeezed) reservoir in a single-mode Fabry-Perot (FP) cavity is presented. McCall's method of handling the standing wave effect is adopted. The study concerns the linear stability analysis of the steady state solutions and the effect of the squeezed reservoir on the regions of stability. The symmetry of the stable regions is broken for some values of the squeezed reservoir.

AMS Subject Classification: 37N20
Key Words: stability analysis, counter-propagating fields, Maxwell-Bloch equations

Received: July 10, 2002

1. Introduction

Non-linear dynamical systems, in addition to their fundamental aspect in the field of mathematics and their applications, have direct potential technological applications. In recent years, research has focused on systems with external feedback (e.g. see [1] and therein). This feedback has played a crucial role in the generation of instabilities in these systems [1, 2] (which also occur when there is no external feedback [3]). Due to the non-linear coupling of matter (e.g. atoms) with radiation, the study of optical bistability (OB) is widely investigated (see [4, 5]). This phenomenon has potential applications in optical telecommunications and the quantum processing of information (see [6, 7]).

On the other hand, the experimental realisation of the squeezed states of the radiation field of reduced quantum fluctuation (see recent review article [8]) has encouraged the investigation of bistable behaviour among other non-linear systems in such states as stability analysis, chaos and bifurcation [9]. Few numerical studies have been performed that indicate the presence of chaotic dynamics in non-dissipative interaction [10]. Also, in the slowly varying envelope approximation and the rotating wave approximation, the study had covered the unperturbed and perturbed problem [11] of Maxwell-Bloch equations in the ring cavity configuration for a probe laser interacting with a two-level atomic medium material sample.

In this paper the investigation of the interaction of two counter-propagating beams of equal intensity in a two-level atomic medium, Figure 1, where the medium is in contact with a phase-sensitive environment- cf. [9]- (called a squeezed reservoir) is considered. McCall's [12] method for handling the field harmonics has been adopted. This method is based on a spatial average for the harmonics over a short-distance, that is, considering the Bloch atomic harmonic components as constants over this short distance (cf. [13]).

Our investigation generalizes earlier work [14] where the reservoir, responsible for the dissipative process, was taken in the normal vacuum state (no phase-sensitive information).

It is important to note that since the internal feedback arises from the interference between the two beams, the calculations are performed *without the mean-field approximation* in which the fields are replaced by their space averaged values e.g. [2, 9, 13]. The linear stability analysis

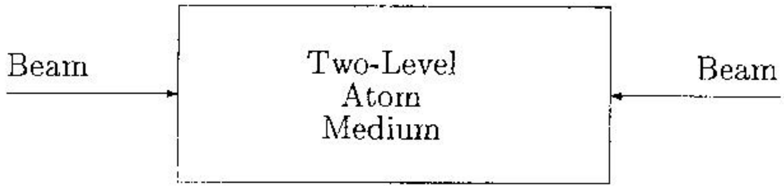

Figure 1: Geometry for interaction of two counter-propagating beams

around the steady state regime is investigated analytically and numerically.

2. The Model Equations

We consider an atomic system in a single mode Fabry-Perot (FP) cavity, Figure 1. The atomic system is composed of two-level atoms (homogeneously broadened) and in contact with a squeezed vacuum reservoir [9] rather than the usual ordinary vacuum reservoir [14]. The model Maxwell-Bloch equations in the plane-wave approximation are of the form [13],

$$\frac{1}{c}\frac{\partial \epsilon_f}{\partial t} + \frac{\partial \epsilon_f}{\partial z} = gP_f \tag{1a}$$

$$\frac{1}{c}\frac{\partial \epsilon_b}{\partial t} + \frac{\partial \epsilon_b}{\partial z} = gP_b \tag{1b}$$

and

$$\begin{aligned} \frac{\partial J_-}{\partial t} + BJ_- &= -2\epsilon J_z - \gamma M J_+ &(2a)\\ \frac{\partial J_+}{\partial t} + B^* J_+ &= -2\epsilon^* J_z - \gamma M^* J_- &(2b)\\ \frac{\partial J_z}{\partial t} + (B + B^*)J_z &= -\frac{\gamma}{2} + (\epsilon^* J_- + \epsilon J_+)\,, &(2c) \end{aligned}$$

where

$$\epsilon = \epsilon_f e^{-ikz} + \epsilon_b e^{ikz}, \tag{3}$$

ϵ_f and ϵ_b are the spatially forward and backward field envelopes respectively, $g = 2\pi n\omega_0\mu/c^2$ is the coupling constant with n the atomic density number, μ is the dipole matrix element, ω_o is the resonant atomic frequency, γ is the decay coefficient, $B = \frac{\gamma}{2}(1+2N) + i\delta$, where $\delta = \omega_o - \omega_L$ is the atomic detuning, ω_L is the (laser) driving field frequency. The squeezed vacuum field parameters, N and M, are the average photon number in the squeezed field and the degree of squeezing respectively, such that $|M|^2 \leq N(N+1)$ [9]. $J_\pm$ are the quadrature components of the mean atomic polarization and J_z is the atomic inversion per atom. If $N = M = 0$, the system of model equations describes the normal vacuum case [14].

The formal steady state solutions, where $\frac{\partial J_-}{\partial t} = \frac{\partial J_+}{\partial t} = \frac{\partial J_z}{\partial t} = 0$, of the Bloch equations (2) are given by

$$J_z^{st} = \frac{-(1+\delta^2)}{2(1+2N)} \left[1 + \delta^2 + b_1|\epsilon|^2\right]^{-1}, \tag{4a}$$

$$J_-^{st} = \frac{-1}{\sqrt{2}} \epsilon\,(b_1 - ib_2) \left[1 + \delta^2 + b_1|\epsilon|^2\right]^{-1}, \tag{4b}$$

where

$$b_1 = 1 - \frac{2|M|\cos\phi}{1+2N},$$

$$b_2 = \frac{\delta + 2|M|\sin\phi}{1+2N}.$$

3. McCall Treatment of the Standing Waves

We use McCall treatment [12] for the standing (plane) wave effects due to oppositely propagating fields inside the FP cavity. This method is based on a spatial average for the harmonics over a short distance (π/k; k is an integer), i. e. by considering the Bloch harmonics for the energy and polarization components as constants over this short distance.

In the steady state

$$J_z^{st} = \frac{-(1+\delta^2)\rho}{2(1+2N)\epsilon_f\epsilon_b^*} (\rho - \rho_1)^{-1}(\rho - \rho_2)^{-1}, \tag{5a}$$

where $\rho = e^{-2ik_o z}$, and

$$\rho_{\frac{1}{2}} = \frac{1}{2\epsilon_f \epsilon_b^*} \Big[-[1 + \delta^2 + b_1(|\epsilon_f|^2 + |\epsilon_b|^2)] \pm \sqrt{[1 + \delta^2 + b_1(|\epsilon_f|^2 + |\epsilon_b|^2)]^2 - 4|\epsilon_f|^2|\epsilon_b|^2} \Big] . \quad (5b)$$

Consistent with eq. (3) we expand J_z^{st} as Fourier series:

$$J_z^{st} = \frac{1}{2} \sum_{n=0}^{\infty} J_{z,n}(e^{2nik_o z} + e^{-2ink_o z}) , \quad (6)$$

where

$$J_{z,n} = \frac{1}{2L} \oint_o^L J_z^{st}(e^{2nik_o z} + e^{-2ink_o z}) dz . \quad (7)$$

For the non-harmonic components ($n = 0$) and by using eq-s (5), a linear transformation is applied on z so that $\tilde{z} = -2ik_o z$, where $L = \pi/k_o$ and hence the 'clockwise' integration in (7) with respect to ρ gives

$$J_{z,0} = \frac{-(1+\delta^2)}{2(1+2N)} \left\{ [1 + \delta^2 + b_1(|\epsilon_f|^2 + |\epsilon_b|^2)]^2 - 4|\epsilon_f|^2|\epsilon_b|^2 \right\}^{-\frac{1}{2}} . \quad (8)$$

Similarly, the polarization is expanded as [13],

$$J_-(z) = \sum_{n=1}^{\infty} P_n(z)\, e^{i(2n-1)k_o z} + \sum_{n=-\infty}^{-1} P_n(z)\, e^{i(2n-1)k_o z} = (J_+(z))^* , \quad (9)$$

and the first harmonics $P_f \equiv P_{-1}$ and $P_b \equiv P_1$, are calculated as Fourier integrals:

$$P_{f,b} = \frac{k}{\pi} \int_0^{\pi/k} J_-(z) e^{\pm ik_o z} dz . \quad (9a)$$

In the steady state, and from eq-s (2a, b), $J_-(z)$ is obtained in the form [13]:

$$J_-(z) = \gamma \frac{n_1 e^{-ik_o z} + n_2 e^{ik_o z}}{d_1 + (d_2 e^{2ik_o z} + c.c.)} , \quad (10)$$

where

$$\begin{aligned} n_1 &= \gamma M\epsilon_b^* - B^*\epsilon_f, \\ n_2 &= \gamma M\epsilon_f^* - B^*\epsilon_b, \\ d_1 &= (B+B^*)(\gamma^2 \mid M \mid^2 - \mid B \mid^2 -2(\mid \epsilon_f \mid^2 + \mid \epsilon_b \mid^2)) \\ &\quad +4\gamma(M\epsilon_f^*\epsilon_b^* + M^*\epsilon_f\epsilon_b), \\ d_2 &= 2\gamma(M(\epsilon_f^*)^2 + M^*\epsilon_b^2) - 2(B+B^*)\epsilon_f^*\epsilon_b. \end{aligned} \tag{11}$$

By substituting eq. (10) into the expansion of $J_\pm(z)$ and evaluating the integrals [13], we get

$$P_f = \frac{\gamma}{2d_2}\left[n_2 + \frac{2n_1d_2 - n_2d_1}{\sqrt{d_1^2 - 4 \mid d_2 \mid^2}}\right], \tag{12a}$$

$$P_b = \frac{\gamma}{2d_2^*}\left[n_1 + \frac{2n_2d_2^* - n_1d_1}{\sqrt{d_1^2 - 4 \mid d_2 \mid^2}}\right]. \tag{12b}$$

4. Steady State Maxwell's Equations

By setting the time derivatives $\partial\epsilon_{f,b}/\partial t = 0$, and using (3), then Maxwell equations (1a, b) take the following form:

$$\begin{aligned} \frac{d|\epsilon_f|}{dz} &= \frac{g\gamma}{2}(\beta_3^2+\beta_4^2)^{-1}\left[\beta_3(\beta_6+\frac{\beta_1}{\beta_5}) + \beta_4(\beta_7+\frac{\beta_2}{\beta_5})\right] \\ \frac{d|\epsilon_b|}{dz} &= \frac{-g\gamma}{2}(\beta_3^2+\beta_4^2)^{-1}\left[\beta_3(\beta_8+\frac{\Omega_1}{\beta_5}) - \beta_4(\beta_9+\frac{\Omega_2}{\beta_5})\right] \\ \frac{d\phi_f}{dz} &= \frac{g\gamma}{2} \mid \epsilon_f \mid^{-1} (\beta_3^2+\beta_4^2)^{-1}\left[\beta_3(\beta_7+\frac{\beta_2}{\beta_5}) - \beta_4(\beta_6+\frac{\beta_1}{\beta_5})\right] \\ \frac{d\phi_b}{dz} &= \frac{-g\gamma}{2} \mid \epsilon_b \mid^{-1} (\beta_3^2+\beta_4^2)^{-1}\left[\beta_3(\beta_9+\frac{\Omega_2}{\beta_5}) + \beta_4(\beta_8+\frac{\Omega_1}{\beta_5})\right], \end{aligned} \tag{13}$$

where

$$
\begin{aligned}
M &= |M| e^{i\phi_m}, \\
\epsilon_f &= |\epsilon_f| e^{i\phi_f}, \\
\epsilon_b &= |\epsilon_b| e^{i\phi_b}, \\
\phi &= \phi_m - (\phi_b + \phi_f) \\
\beta_1 &= 4 |M| \alpha_1 + \cos(\phi)\alpha_3 + \frac{1}{2}(1+2N)\alpha_4 \\
\beta_2 &= 4 |M| \alpha_2 + \sin(\phi)\alpha_3 - \delta\alpha_4 \\
\beta_3 &= 2 |M| \cos(\phi)(|\epsilon_f|^2 + |\epsilon_b|^2) - 2(1+2N) |\epsilon_f||\epsilon_b| \\
\beta_4 &= 2 |M| \sin(\phi)(|\epsilon_f|^2 - |\epsilon_b|^2) \\
\beta_5 &= \{ [(1+2N)[\gamma^2 |M|^2 - |B|^2 - 2(|\epsilon_f|^2 + |\epsilon_b|^2)] \\
&\quad +8 |M||\epsilon_f||\epsilon_b| \cos(\phi)]^2 \\
&\quad - 4 [4 |M|^2 [|\epsilon_f|^4 + |\epsilon_b|^4 + 2 |\epsilon_f|^2 |\epsilon_b|^2 \cos(2\phi)] \\
&\quad + 4(1+2N)^2 |\epsilon_f|^2 |\epsilon_b|^2 \\
&\quad - 8 |M| (1+2N) |\epsilon_f||\epsilon_b| [|\epsilon_f|^2 + |\epsilon_b|^2] \cos(\phi)] \}^{\frac{1}{2}} \\
\beta_6 &= |M||\epsilon_f| \cos(\phi) - \frac{1}{2}(1+2N) |\epsilon_b| \\
\beta_7 &= |M||\epsilon_f| \sin(\phi) + \frac{\delta}{\gamma} |\epsilon_b| \\
\beta_8 &= |M||\epsilon_b| \cos(\phi) - \frac{1}{2}(1+2N) |\epsilon_b| \\
\beta_9 &= |M||\epsilon_b| \sin(\phi) + \frac{\delta}{\gamma} |\epsilon_b|
\end{aligned}
$$

$$
\begin{aligned}
\Omega_1 &= 4 \mid M \mid A_1 + A_3 \cos(\phi) + \frac{\gamma}{2}(1+2N) A_4 \\
\Omega_2 &= 4 \mid M \mid A_2 + A_3 \sin(\phi) - \delta A_4 \\
\alpha_1 &= \mid M \mid\mid \epsilon_b \mid\mid \epsilon_f \mid^2 \cos(\phi) + \mid M \mid\mid \epsilon_b \mid^3 \\
&\quad - \mid \epsilon_f \mid^3 [\frac{1}{2}(1+2N)\cos(\phi) + \frac{\delta}{\gamma}\sin(\phi)] \\
&\quad + \mid \epsilon_f \mid\mid \epsilon_b \mid^2 [\frac{1}{2}(1+2N)\sin(\phi) + \frac{\delta}{\gamma}\cos(\phi)] \\
\alpha_3 &= -4 \mid M \mid (1+2N) \mid \epsilon_f \mid\mid \epsilon_b \mid^2 - 8 \mid M \mid^2 \mid \epsilon_f \mid^2 \mid \epsilon_b \mid \cos(\phi) \\
&\quad - \mid M \mid (1+2N) \mid \epsilon_f \mid [\gamma^2 \mid M \mid^2 - \mid B \mid^2 - 2(\mid \epsilon_f \mid^2 + \mid \epsilon_b \mid^2)] \\
\alpha_4 &= 4(1+2N) \mid \epsilon_f \mid^2 \mid \epsilon_b \mid + 8 \mid M \mid\mid \epsilon_f \mid\mid \epsilon_b \mid^2 \cos(\phi) \\
&\quad + (1+2N) \mid \epsilon_b \mid [\gamma^2 \mid M \mid^2 - \mid B \mid^2 - 2(\mid \epsilon_f \mid^2 + \mid \epsilon_b \mid^2)] \\
A_1 &= \mid M \mid\mid \epsilon_f \mid^3 + \mid M \mid\mid \epsilon_f \mid\mid \epsilon_b \mid^2 \cos(2\phi) \\
&\quad - \mid \epsilon_b \mid\mid \epsilon_f \mid^2 [\frac{1}{2}(1+2N)\cos(\phi) - \frac{\delta}{\gamma}\sin(\phi)] \\
&\quad - \mid \epsilon_b \mid^3 [\frac{1}{2}(1+2N)\cos(\phi) + \frac{\delta}{\gamma}\sin(\phi)] \\
A_2 &= \mid M \mid\mid \epsilon_f \mid\mid \epsilon_b \mid^2 \sin(2\phi) + \mid \epsilon_b \mid\mid \epsilon_f \mid^2 [\frac{1}{2}(1+2N)\sin(\phi) + \frac{\delta}{\gamma}\cos(\phi) \\
&\quad - \mid \epsilon_b \mid^3 [\frac{1}{2}(1+2N)\sin(\phi) - \frac{\delta}{\gamma}\cos(\phi)] \\
A_3 &= -4 \mid M \mid (1+2N) \mid \epsilon_f \mid^2 \mid \epsilon_b \mid - 8 \mid M \mid^2 \mid \epsilon_f \mid\mid \epsilon_b \mid^2 \cos(\phi) \\
&\quad - \mid M \mid (1+2N) \mid \epsilon_b \mid [\gamma^2 \mid M \mid^2 - \mid B \mid^2 - 2(\mid \epsilon_f \mid^2 + \mid \epsilon_b \mid^2)]
\end{aligned}
$$

$$A_4 = 4(1+2N) \mid \epsilon_b \mid^2 \mid \epsilon_f \mid + 8 \mid M \mid \mid \epsilon_f \mid^2 \mid \epsilon_b \mid \cos(\phi)$$
$$+ (1+2N) \mid \epsilon_f \mid [\gamma^2 \mid M \mid^2 - \mid B \mid^2 - 2(\mid \epsilon_f \mid^2 + \mid \epsilon_b \mid^2)] .$$

Within the limits of very large detuning ($\delta \gg 1$) and small absorption ($\alpha_0 L/\delta^2 \ll 1$), we can assume that the two beams are not depleted (not dependent on z), so $d|\epsilon_{f,b}|/dz = 0$. Thus, we consider the phase changes only as the beams propagate along the medium. So,

$$\frac{d\phi_f}{dz} = \frac{g\gamma}{2\sqrt{I}}(\beta_3)^{-1}(\beta_7 + \frac{\beta_2}{\beta_5}) = \theta ,$$
$$\frac{d\phi_b}{dz} = -\frac{g\gamma}{2\sqrt{I}}(\beta_3)^{-1}(\beta_9 + \frac{\beta_2}{\beta_5}) = -\theta , \qquad (14)$$

where $\mid \epsilon_{f,b} \mid^2 = I$. Hence, from (3) and (14) and within the above limits, the steady state field is given by

$$\epsilon_{ss}(z) = \sqrt{I} e^{-i(k_o - \theta)z} + \sqrt{I} e^{i(k_0 - \theta)z} . \qquad (15)$$

5. Linear Stability Analysis

Adding small perturbations to the steady state values ϵ_{ss}, $r_0(= J_z^{st})$ and $r_{\pm,0}(= J_{\pm}^{st})$ and taking their time-dependence to be $e^{\lambda t}$, the real part of λ represents the growth rate of perturbation, while the imaginary part of λ represents its oscillation frequency, say Ω, near the instability threshold. Note that a field oscillating at $\omega + \Omega$ may interact with the strong field at ω to generate a field at $\omega - \Omega$ [14]. We, therefore, write the perturbations as a sum of two terms oscillating at $\pm\Omega$ and denote their amplitudes with super-script + and − respectively,

$$J_z(z,t) = r_o(z) + r_3^+(z)e^{\lambda t} + r_3^-(z)e^{\lambda^* t} \tag{16a}$$

$$J_-(z,t) = r_{-,o}(z) + r^+(z)e^{\lambda t} + r^-(z)e^{\lambda^* t} \tag{16b}$$

$$J_+(z,t) = (J_-(z,t))^* = r^*_{-,o}(z) + (r^+(z))^* e^{\lambda^* t} + (r^-(z))^* e^{\lambda t} \tag{16c}$$

$$\epsilon(z,t) = \epsilon_{ss}(z) + \epsilon^+(z)e^{\lambda t} + \epsilon^-(z)e^{\lambda^* t} \tag{16d}$$

$$\epsilon^*(z,t) = \epsilon^*_{ss}(z) + (\epsilon^+(z))^* e^{\lambda^* t} + (\epsilon^-(z))^* e^{\lambda t}. \tag{16e}$$

On differentiating (16a) with respect to t and comparing the coefficients of $e^{\lambda t}$ and $e^{\lambda^* t}$ we get,

$$(\lambda + B + B^*)r_3^+ = \epsilon_{ss}(r^-)^* + \epsilon^*_{ss}r^+ + \epsilon^+(r_{-,o})^* + (\epsilon^-)^* r_{-,o}, \tag{17a}$$

$$(\lambda^* + B + B^*)r_3^- = \epsilon_{ss}(r^+)^* + \epsilon^*_{ss}r^- + \epsilon^-(r_{-,o})^* + (\epsilon^+)^* r_{-,o}, \tag{17b}$$

where $r_{\pm,3} = r_{\pm,3}(z)$, and $\epsilon_{ss} = \epsilon_{ss}(z)$. For the non-harmonic component in (17), we replace the terms of

$$\epsilon^+(r_{-,o})^* \longrightarrow \left\{ \epsilon^+ \frac{-2r_o(\epsilon^*_{ss}B - \gamma\epsilon_{ss}M^*)}{|B|^2 - \gamma^2|M|^2} \right\}$$

and

$$\epsilon^- r_{-,o} \longrightarrow \left\{ (\epsilon^-)^* \frac{-2r_o(\epsilon_{ss}B^* - \gamma\epsilon^*_{ss}M)}{|B|^2 - \gamma^2|M|^2} \right\}, \tag{18}$$

where $r_o = r_{3,0}$ given in (7).

Thus with (18) substituted in (17a, b) we get

$$(\lambda + B + B^*)r_3^+ = \epsilon_{ss}(r^-)^* + \epsilon^*_{ss}r^+ - \epsilon^+ \frac{2r_o(\epsilon^*_{ss}B - \gamma\epsilon_{ss}M^*)}{|B|^2 - \gamma^2|M|^2} - (\epsilon^-)^* \frac{2r_o(\epsilon_{ss}B^* - \gamma\epsilon^*_{ss}M)}{|B|^2 - \gamma^2|M|^2}. \tag{19}$$

To get r^- and r^+ in a similar manner we use eq-s (2a, b) and (16b, c), then comparing the coefficients of $e^{\lambda t}$ and $e^{\lambda^* t}$ and solving for r^+ and r^- in terms of r_o and $r_3^\pm$ where $(r_3^-)^* = r_3^+$, we get

$$r^+ = 2r_3^+ \frac{[\gamma M\epsilon_{ss}^* - \epsilon_{ss}(\lambda + B^*)]}{(\lambda + B)(\lambda + B^*) - \gamma^2|M|^2} + 2r_o \frac{[\gamma M(\epsilon^-)^* - \epsilon^+(\lambda + B^*)]}{(\lambda + B)(\lambda + B^*) - \gamma^2|M|^2}, \quad (20a)$$

$$(r^-)^* = \frac{-2r_3^+}{\lambda + B^*}\left\{\frac{\gamma M^*[\gamma M\epsilon_{ss}^* - \epsilon_{ss}(\lambda + B^*)]}{(\lambda + B)(\lambda + B^*) - \gamma^2|M|^2} + \epsilon_{ss}^*\right\} - \frac{2r_o}{\lambda + B^*}\left\{\frac{\gamma M^*[\gamma M(\epsilon^-)^* - \epsilon^+(\lambda + B^*)]}{(\lambda + B)(\lambda + B^*) - \gamma^2|M|^2} + (\epsilon^-)^*\right\}. \quad (20b)$$

Using eq-s (20) into (17a) we get

$$r_3^+ = \frac{\tilde{A}_1}{\tilde{A}_2} r_o, \quad (21)$$

where

$$\tilde{A}_1 = -2\epsilon_{ss}\left\{\gamma M^*[\gamma M(\epsilon^-)^* - \epsilon^+(\lambda + B^*)] + \eta(\epsilon^-)^*\right\} + 2\epsilon_{ss}^*(\lambda + B^*)[\gamma M(\epsilon^-)^* - \epsilon^+(\lambda + B^*)] - \frac{2\eta(\lambda + B^*)}{|B|^2 - \gamma^2|M|^2}\left\{\epsilon^+[\epsilon_{ss}^* B - \gamma\epsilon_{ss}M^*] + (\epsilon^-)^*[\epsilon_{ss}B^* - \gamma\epsilon_{ss}^* M]\right\} \quad (21a)$$

$$\tilde{A}_2 = \eta(\lambda + B^*)(\lambda + B + B^*) + 2\gamma M^*\epsilon_{ss}[\gamma M\epsilon_{ss}^* - \epsilon_{ss}(\lambda + B^*)] + 2\epsilon_{ss}\epsilon_{ss}^*\eta + 2\epsilon_{ss}^*(\lambda + B^*)[\gamma M\epsilon_{ss}^* - \epsilon_{ss}(\lambda + B^*)] \quad (21b)$$

$$\eta = (\lambda + B)(\lambda + B^*) - \gamma^2|M|^2. \quad (21c)$$

By redefining the fields

$$\epsilon^+ = F_1 e^{-ikz} + F_2 e^{ikz}$$

$$\epsilon^- = F_3^* e^{-ikz} + F_4^* e^{ikz}, \quad (22)$$

and since,

$$\epsilon(z,t) \equiv \epsilon_{ss} + \epsilon^{+} e^{\lambda t} + \epsilon^{-} e^{\lambda^* t} \tag{23}$$

we get

$$\begin{aligned} \epsilon_f(z,t) &= e^{i\theta z}(\sqrt{I} + F_1(z)e^{\lambda t} + F_3^*(z)e^{\lambda^* t}) \\ \epsilon_b(z,t) &= e^{-i\theta z}(\sqrt{I} + F_2(z)e^{\lambda t} + F_4^*(z)e^{\lambda^* t}). \end{aligned} \tag{24}$$

By using (24) into eq-s (1) and consistently putting $P_{f,b}$ in the form

$$\begin{aligned} P_f &= P_f^{st} + G_1 e^{\lambda t} + G_3^* e^{\lambda^* t} \\ P_b &= P_b^{st} + G_2 e^{\lambda t} + G_4^* e^{\lambda^* t}, \end{aligned} \tag{25}$$

where the G_i's are the phase-matched polarizations obtained by Fourier transformation of $r_\pm$ (see Appendix A) and $k = k_0 - \theta$ and equating the coefficients of $e^{\lambda t}$ and $e^{\lambda^* t}$, one gets a system of differential equations written in a matrix form:

$$\frac{d}{dz}\bar{F} = \bar{M}_1 \bar{F}, \tag{26}$$

where $\bar{F} = (F_1, F_2, F_3, F_4)$. The elements of the matrix $\bar{M}_1$ represent phase-matched four-wave-mixing interactions which couple each F_i to the others [14] (see Appendix B). The formal solution of eq. (26) is given by [14]

$$F(z) = \sum_{i=1}^{4} c_i \bar{f}_i e^{\sigma_i z}, \tag{27}$$

where the $\bar{f}_i$'s and σ_i's are the eigenvectors and eigenvalues of the matrix $\bar{M}_1$, and the c_i's are the coefficients to be determined by the boundary conditions (BC). These BC for $F_{1,3}$ as forward amplitudes at $z = 0$ and $F_{2,4}$ as backward amplitudes at $z = L$, are:

$$F_1(0) = F_3(0) = F_2(L) = F_4(L) = 0, \tag{28}$$

i.e. the perturbation fields are zero at the inputs. Using the BC, eq. (28), into (27) we get the system

$$\bar{C}\bar{A} = 0, \tag{29}$$

where $\bar{C} = (c_1, c_2, c_3, c_4)$ and $\bar{A}$ are defined in Appendix B.

The solutions (27) with (29) are then non-zero (non-trivial) only if $det(\bar{A}) = 0$. This requirement is achieved by finding the threshold of instability and setting $Re(\lambda) = 0$, which describes the transition from stable $[Re(\lambda) < 0]$ to unstable $[Re(\lambda) > 0]$.

In our study of the behaviour of the system at the instability threshold $[Re(\lambda) = 0]$, we had two unknowns for different values of the squeezed parameters N and ϕ_m so that $det(\bar{A}(\lambda = i\alpha_n)) = 0$. Note that the detuning is large according to our assumption ($\delta \gg 1$) and the cooperative parameter C satisfies $C \ll \delta^2$ in order to be consistent with the undepleted fields.

We now comment on the numerical results. For the normal vacuum case when $N = \phi = 0$, Figure 2, a "Top-like" region of transition appears for large positive values of α_n (i. e. far off-resonance) and a "Whip-like" small region appears for the negative values of α_n. Both regions are symmetric with respect to θ. In the squeezed vacuum case with $N = 0.1$ and, for different values of ϕ, the transition regions ("Top" and "Whip") increase for $\phi = 0$ and keep their symmetry with respect to θ (Figure 3a). For $\phi = \pi/2$, the "Top" region gets larger and asymmetric with respect to θ, with scattered regions around the "Whip" (Figure 3b). The case of $\phi = \pi$ (Figure 3c) is similar to the normal vacuum case (Figure 2). For a larger value of $N(= 1)$, Figure 4, there is a tendency of the splitting of the "Top" region and the threshold regions for non-zero values of θ and the "Whip" region almost vanishes. For a further increase of $N(= 5)$ and for $\phi = \pi$ (Figure 5) the "Whip" region disappears completely and the "Top" region is seperated into stretched isles in the positive θ-direction and with large values of α_n.

Acknowledgment

The author is grateful to the University Research Committee of King Abdul Aziz University for their financial support (Project N. 418504).

References

[1] *Instabilities in Active Media* (Ed-s: N.B. Abraham, L.A. Lugiato, L.M. Narducci), J. Opt. Soc. Am. B **2** (1985); *Optical Instabilities* (Ed-s: R.W. Boyd, M.G. Raymer, L.M. Narducci), Cambridge University Press, Cambridge (1986).

[2] H.A. Batarfi, *Analytical and Numerical Study of Nonlinear Phenomenon in Squeezed Vacua*, Ph.D. Thesis, University of Manchester (1995).

[3] Y. Silberberg, I. Bar-Joseph, *Phys. Rev. Lett.*, **48**, No 1541 (1982); *J. Opt. Soc. Am. B*, **1**, No 662 (1984).

[4] H. Haken, *Synergetics*, Springer (1977).

[5] I. Prigogine, G. Nicolis, *Self Organisation in Non-equilibrium Systems*, Wiley, N.Y. (1977).

[6] D. Smith, D. Miller, Computing at the speed of light, *New Scientist* (Feb. 21, 1980), 554-556.

[7] J.N. Lee (Ed.), *Design Issues in Optical Preocessing*, Cambridge Univ. Press, U.K. (1995).

[8] V.V. Dodonov, *J. Opt. B: Quantum Semiclass. Opt.*, **4**, No R1 (2002).

[9] S.S. Hassan, H.A. Batarfi, R. Saunders, R.K. Bullough, *Europ. Phys. J. D*, **8**, No 403 (2000); H.A. Batarfi, S.S. Hassan, R. Saunders, R.K. Bullough, *Europ. Phys. J. D*, **8**, No 417 (2000).

[10] M.C. Gutzwiller, B.B. Mandelbrot, *Phys. Rev. Lett.*, **60**, No 673 (1988).

[11] D.D. Holm, G. Kovacic, *Chaos in Australia* (Ed-s: G. Brown, A. Opie), 57-94, World Scientific Publishing, Singapore (1993).

[12] S.L. McCall, *Phys. Rev. A*, **9**, No 1515 (1974); S.L. McCall, H.M. Gibbs, *Opt. Comm.*, **33**, No 335 (1980).

[13] S.M.A. Maize, M.F. Ali, S.S. Hassan, *Nonlinear Optics*, **8**, No 219 (1994).

[14] I. Bar-Joseph, Y. Silberberg, *Phys. Rev. A*, **36**, No 1731 (1987).

Appendix A: Evaluation of The Phase-Matched Polarizations $G_{i=1,2,3,4}$

We have

$$G_{1,2} = \frac{1}{\pi} \int_o^\pi r^+ (\frac{x}{k})\, e^{\pm ix}\, dx \qquad \text{(A. 1a)}$$

and

$$G^*_{3,4} = \frac{1}{\pi} \int_o^\pi r^- (\frac{x}{\pi})\, e^{\pm ix}\, dx. \qquad \text{(A. 1b)}$$

First, we calculate G_1. From eq-s (20a), (21) and (22), eq. (A. 1a)

$$G_1 = \frac{2}{\eta\pi} \left[\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 \right], \qquad \text{(A. 2)}$$

where

$$\tilde{I}_1 = \int_0^\pi \gamma\, M\, r_{-,o}\, (\epsilon^-)^*\, e^{\pm ix}\, dx\,, \qquad \text{(A. 3a)}$$

$$\tilde{I}_2 = \int_0^\pi (\lambda + B^*)\, r_{-,o}\, \epsilon^+\, e^{\pm ix}\, dx \qquad \text{(A. 3b)}$$

$$\tilde{I}_3 = \int_0^\pi (\gamma M - (\lambda + B^*))\, r_{-,o}\, \epsilon_{ss}\, \frac{\tilde{A}_1}{\tilde{A}_2}\, e^{\pm ix}\, dx\,. \qquad \text{(A. 3c)}$$

Now, from eq-s (18) and (22) we have

$$r_{-,o}\, (\epsilon^-)^* = -2\sqrt{I}\, r_o\, \frac{B^* - \gamma M}{|B|^2 - \gamma^2 |M|^2} \left[F_3\, e^{2ikz} + F_4\, e^{-2ikz} + \, (F_3 + F_4) \right], \qquad \text{(A.4a)}$$

$$r_{-,o}\, \epsilon^+ = -2\sqrt{I}\, r_o\, \frac{B^* - \gamma M}{|B|^2 - \gamma^2 |M|^2} \left[F_2\, e^{2ikz} + F_1\, e^{-2ikz} + \, (F_1 + F_2) \right], \qquad \text{(A.4b)}$$

$$r_{-,o}\,\epsilon_{ss} \quad = \quad -2\,I\,r_o\,\frac{B^* - \gamma M}{|B|^2 - \gamma^2|M|^2}\left[e^{2ikz} + e^{-2ikz} + 2\right]. \qquad \text{(A.4c)}$$

Similarly, from (21a, b, c) and (22) we put,

$$\tilde{A}_1 \quad = \quad 2\sqrt{I}\left(a_1\,e^{2ikz} + a_2\,e^{-2ikz} + (a_1 + a_2)\right), \qquad \text{(A. 5a)}$$

$$\tilde{A}_2 \quad = \quad a_3\left(e^{2ikz} + e^{-2ikz}\right) + a_4\,, \qquad \text{(A. 5b)}$$

where

$$a_1 = F_3\left[(\lambda + B^* - \gamma\,M^*)\,\gamma M - \eta - \frac{\eta(\lambda + B^*)}{|B|^2 - \gamma^2|M|^2}(B^* - \gamma\,M)\right] + F_2\left[-(\lambda + B^* - \gamma\,M^*)\,(\lambda + B^*) - \frac{\eta(\lambda + B^*)}{|B|^2 - \gamma^2|M|^2}(B - \gamma\,M^*)\right]$$

$$a_2 = F_4\left[(\lambda + B^* - \gamma\,M^*)\,\gamma M - \eta - \frac{\eta(\lambda + B^*)}{|B|^2 - \gamma^2|M|^2}(B^* - \gamma\,M)\right] + F_1\left[-(\lambda + B^* - \gamma\,M^*)\,(\lambda + B^*) - \frac{\eta(\lambda + B^*)}{|B|^2 - \gamma^2|M|^2}(B - \gamma\,M^*)\right]$$

$$a_3 \quad = \quad 2\,I\,[\eta + (\gamma M - (\lambda + B^*))(\gamma\,M^* + \lambda + B^*)]$$

$$a_4 \quad = \quad 2\,a_3 + \eta(\lambda + B^*)(\lambda + B + B^*). \qquad \text{(A. 6)}$$

First, to calculate $\tilde{I}_1$ of (A. 3a), we use (A.4a) to get

$$\tilde{I}_1 = -\frac{16i}{3}\sqrt{I}\,\gamma M\,r_o\,\frac{B^* - \gamma\,M}{|B|^2 - \gamma^2|M|^2}F_3\,, \qquad \text{(A. 7)}$$

where (consistent with the McCall method) $F_i(z)$, i=1, ..., 4 are considered constants over the integration period $[0, \pi]$.

For $\tilde{I}_2$, eq. (A. 3b), we use (A.4b) to get

$$\tilde{I}_2 = \frac{16i}{3}\sqrt{I}\,(\lambda + B^*)\,r_o\,\frac{B^* - \gamma\,M}{|B|^2 - \gamma^2|M|^2}F_2\,. \qquad \text{(A. 8)}$$

Similarly, for $\tilde{I}_3$, eq. (A. 3c), by using eq-s (A.4c), and (A. 5), we get

$$\tilde{I}_3 = \alpha_3 \int_o^{\pi} \left(a_1 e^{5ix} + 3(a_1+a_2)\, e^{ix} + (3a_1+a_2)\, e^{3ix} + a_2\, e^{-3ix} \right.$$
$$\left. + (3a_2 + a_1)\, e^{-ix}\right) \left(2a_3 \cos(2x) + a_4\right)^{-1} dx\,, \quad \text{(A. 9a)}$$

where

$$\alpha_3 = -4I\sqrt{I}\, r_o(\gamma M - (\lambda + B^*)) \frac{B^* - \gamma M}{|B|^2 - \gamma^2 |M|^2}\,. \quad \text{(A. 9b)}$$

In eq. (A. 9a) we need to evaluate the integral of the form

$$J(\alpha) = \int_o^{\pi} \frac{e^{i\alpha x}}{a\,\cos(2x) + b} dx\,, \qquad \alpha = 0, \pm 1, \pm 3, \pm 5 \quad \text{(A. 10)}$$

(i) For $\alpha = 0$

$$J(0) = \int_o^{\pi} \frac{dx}{a\,\cos(2x) + b} = \frac{\pi}{\sqrt{b^2 - a^2}}, \quad b^2 > a^2. \quad \text{(A. 11a)}$$

Alternatively, we put $a\,\cos(2x) + b = \tilde{a}\cos^2(x) + \tilde{b}$, where $\tilde{a} = 2a$ and $\tilde{b} = b - a$, which means that

$$J(0) = \int_o^{\pi} \frac{dx}{\tilde{a}\cos^2(x) + \tilde{b}}$$
$$= \frac{\pi}{\sqrt{(\tilde{b} - \frac{1}{2}\tilde{a})^2 - (\frac{1}{2}\tilde{a})^2}}\,;\ \tilde{b}^2 > \tilde{a}^2. \quad \text{(A. 11b)}$$

(ii) For $\alpha = \pm 1$;

$$J(\pm 1) = \pm \frac{2i}{\sqrt{2a(b-a)}}\ \tan^{-1}\left(\sqrt{\frac{2a}{b-a}}\right) \quad \text{(A. 12)}$$

(iii) For $\alpha = \pm 3$; we use $e^{\pm 3ix} = 4\cos^2(x)[\cos(x) \pm i\sin(x)]$-$[3\cos(x) \pm i\sin(x)]$ and eq-s (A. 9b), (A. 10) and (A. 11), to get

$$J(\pm 3) = \pm\frac{4i}{a} \mp \frac{2i}{\sqrt{2a(b-a)}}\left[\frac{2(b-a)}{a} + 1\right]$$
$$\times \tan^{-1}\left(\sqrt{\frac{2a}{b-a}}\right) \quad \text{(A. 13)}$$

(iv) For $\alpha = \pm 5$; similarly,

$$J(5) = \frac{4i}{\tilde{a}}\left(-\frac{10}{3} - \frac{8\tilde{b}}{\tilde{a}}\right) + \frac{2i}{\tilde{a}}\sqrt{\frac{\tilde{a}}{\tilde{b}}}\left[16\frac{\tilde{b}^2}{\tilde{a}^2} + 12\frac{\tilde{b}}{\tilde{a}} + 1\right] \tan^{-1}\left(\sqrt{\frac{\tilde{a}}{\tilde{b}}}\right) \quad \text{(A. 14a}$$

and

$$J(-5) = \frac{i}{a}\left[\frac{20}{3} + \frac{8(b-a)}{a}\right] - \frac{2i}{\sqrt{2a(b-a)}}\left[4\frac{(b-a)^2}{a^2} + 6\frac{b-a}{a} + 1\right]\tan^{-1}\left(\sqrt{\frac{2a}{b-a}}\right).$$

Hence, from (A. 9a), (A. 11), (A. 13) and (A. 14) we get

$$\tilde{I}_3 = \alpha_3\left\{2i\left[\frac{16}{3} - \frac{8}{a}(b-a)\right] + \frac{16\,i\,(b-a)^2}{a\sqrt{2a(b-a)}}\tan^{-1}\left(\sqrt{\frac{2\,a}{b-a}}\right)\right\}. \quad \text{(A. 15)}$$

Substituting (A. 7), (A. 8) and (A. 15) into (A. 2), we get

$$G_1 = \frac{2}{\eta\pi}\left[A_{11}\,F_1 + A_{12}\,F_2 + A_{13}\,F_3 + A_{14}\,F_4\right]. \quad \text{(A. 16a)}$$

The calculations for $G_{2,3,4}$ are very similar and yield the following:

$$G_2 = \frac{2}{\eta\pi}\left[A_{21}\,F_1 + A_{11}\,F_2 + A_{14}\,F_3 + A_{24}\,F_4\right] \quad \text{(A. 16b)}$$

$$G_3 = \frac{2}{\pi}\left[A_{31}\,F_1 + A_{32}\,F_2 + A_{33}\,F_3 + A_{34}\,F_4\right] \quad \text{(A. 16c)}$$

$$G_4 = \frac{2}{\pi}\left[A_{32}\,F_1 + A_{42}\,F_2 + A_{43}\,F_3 + A_{33}\,F_4\right]. \quad \text{(A. 16d)}$$

The coefficients A_{ij} in eq-s (A. 16) are given by,

$$
\begin{aligned}
A_{11} &= 0 \\
A_{12} &= (\lambda + B^*)\, a_{13} + a_{14}\, a_{12} \\
A_{13} &= \gamma M\, a_{13} + a_{14}\, a_{11} \\
A_{14} &= 0 \\
A_{21} &= (\lambda + B^*)\, a_{13} - a_{14}\, a_{12} \\
A_{24} &= -\gamma M\, a_{13} - a_{14}\, a_{11} \\
A_{31} &= \frac{1}{\lambda + B^*}\left[A_{21}\eta_1 + \frac{\lambda + B^*}{\gamma M - \lambda - B^*} a_{13}\right] \\
A_{32} &= \frac{A_{11}}{\lambda + B^*}\eta_1 \\
A_{33} &= \frac{A_{14}}{\lambda + B^*}\eta_1 \\
A_{34} &= \frac{1}{\lambda + B^*}\left[A_{24}\eta_1 - \left(\frac{\gamma M}{\gamma M - \lambda - B^*} - 1\right) a_{13}\right] \\
A_{42} &= \frac{A_{12}\eta_1}{\lambda + B^*} + \frac{a_{13}}{\gamma M - \lambda - B^*} \\
A_{43} &= \frac{1}{\lambda + B^*}\left[A_{13}\eta_1 + \left(\frac{\gamma M}{\gamma M - \lambda - B^*} - 1\right) a_{13}\right] \\
\eta_1 &= -\frac{\gamma}{\eta} M^* - \frac{1}{\gamma M - \lambda - B^*} \\
a_{11} &= (\lambda + B^* - \gamma M^*)\gamma M - \eta - \eta \frac{\lambda + B^*}{|B|^2 - \gamma^2 |M|^2}(B^* - \gamma M) \\
a_{12} &= -(\lambda + B^* - \gamma M^*)(\lambda + B^*) - \eta \frac{\lambda + B^*}{|B|^2 - \gamma^2 |M|^2}(B^* - \gamma M)
\end{aligned}
$$

$$a_{13} = -\frac{16i}{3}\sqrt{I}\, r_o \frac{B^* - \gamma M}{|B|^2 - \gamma^2 |M|^2}$$

$$a_{14} = \alpha_3 \left\{ \frac{8i}{a}\left[\frac{2}{3} - \frac{(b-a)}{a}\right] + \frac{8i(b-a)^2}{a^2\sqrt{2a(b-a)}} \tan^{-1}\left(\sqrt{\frac{2a}{b-a}}\right)\right\} . \quad \text{(A. 17)}$$

Appendix B

In solving the system of differential equations, eq-s (26), the matrix $\bar{M}_1$ is given by,

$$\bar{M}_1 = [C_{ij}] \,; \qquad i, j = 1, 2, 3, 4 \,, \qquad \text{(B. 1)}$$

where

$$C_{11} = \frac{2g}{\eta\pi} A_{11} - (\frac{\lambda}{c} + i\theta)$$

$$C_{12} = \frac{2g}{\eta\pi} A_{12}$$

$$C_{13} = \frac{2g}{\eta\pi} A_{13}$$

$$C_{14} = \frac{2g}{\eta\pi} A_{14}$$

$$C_{21} = \frac{2g}{\eta\pi} A_{21}$$

$$C_{22} = \frac{2g}{\eta\pi} A_{11} - (\frac{\lambda}{c} - i\theta)$$

$$
\begin{aligned}
C_{23} &= \frac{2g}{\eta\pi} A_{14} \\
C_{24} &= \frac{2g}{\eta\pi} A_{24} \\
C_{31} &= \frac{2g}{\pi} A_{31} \\
C_{32} &= \frac{2g}{\pi} A_{32} \\
C_{33} &= \frac{2g}{\pi} A_{33} - (\frac{\lambda}{c} - i\theta) \\
C_{34} &= \frac{2g}{\pi} A_{34} \\
C_{41} &= \frac{2g}{\pi} A_{32} \\
C_{42} &= \frac{2g}{\pi} A_{42} \\
C_{43} &= \frac{2g}{\pi} A_{43} \\
C_{44} &= \frac{2g}{\pi} A_{33} - (\frac{\lambda}{c} + i\theta) \qquad \text{(B. 2)}
\end{aligned}
$$

and the coefficients A_{ij} are given in (A. 17).

Evaluating, numerically, the eigenvalues σ_i and the corresponding eigenvectors $\bar{f}_i = (f_{i1}, f_{i2}, f_{i3}, f_{i4})$, we get the general solution, eq. (27), as:

$$
F(z) = c_1 \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \end{bmatrix} e^{\sigma_1 z} + \ldots + c_4 \begin{bmatrix} f_{41} \\ f_{42} \\ f_{43} \\ f_{44} \end{bmatrix} e^{\sigma_1 z} . \qquad \text{(B. 3)}
$$

Applying the BCs, eqs(28), this system is reduced to the set of homogeneous linear equations:

$$c_1 f_{11} + c_2 f_{21} + c_3 f_{31} + c_4 f_{41} = 0$$

$$c_1 f_{13} + c_2 f_{23} + c_3 f_{33} + c_4 f_{43} = 0$$

$$c_1 f_{12}\, e^{\sigma_1 L} + c_2 f_{22}\, e^{\sigma_2 L} + c_3 f_{32}\, e^{\sigma_3 L} + c_4 f_{42}\, e^{\sigma_4 L} = 0$$

$$c_1 f_{14}\, e^{\sigma_1 L} + c_2 f_{24}\, e^{\sigma_2 L} + c_3 f_{34}\, e^{\sigma_3 L} + c_4 f_{44}\, e^{\sigma_4 L} = 0\,, \qquad \text{(B. 4)}$$

i.e.

$$\bar{C}\,\bar{A} = 0\,, \qquad \text{(B. 5)}$$

where

$$\bar{A} = \begin{bmatrix} f_{11} & f_{12}\, e^{\sigma_1 L} & f_{13} & f_{14}\, e^{\sigma_1 L} \\ f_{21} & f_{22}\, e^{\sigma_2 L} & f_{23} & f_{24}\, e^{\sigma_2 L} \\ f_{31} & f_{32}\, e^{\sigma_3 L} & f_{33} & f_{34}\, e^{\sigma_3 L} \\ f_{41} & f_{42}\, e^{\sigma_4 L} & f_{43} & f_{44}\, e^{\sigma_4 L} \end{bmatrix} \qquad \text{(B. 6)}$$

and

$$\bar{C} = [c_1 \quad c_2 \quad c_3 \quad c_4]\,. \qquad \text{(B. 7)}$$

Figures

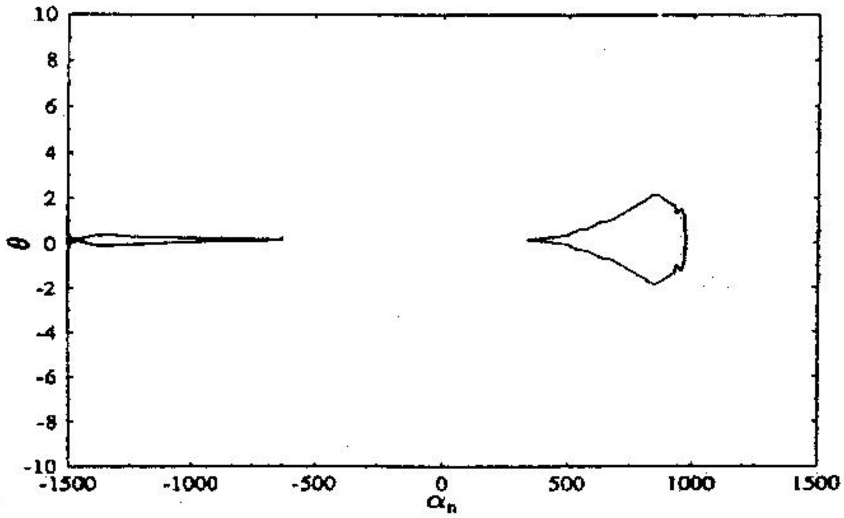

Figure 2: α_n versus θ for the case of $N = \phi_m = 0$ (normal case)

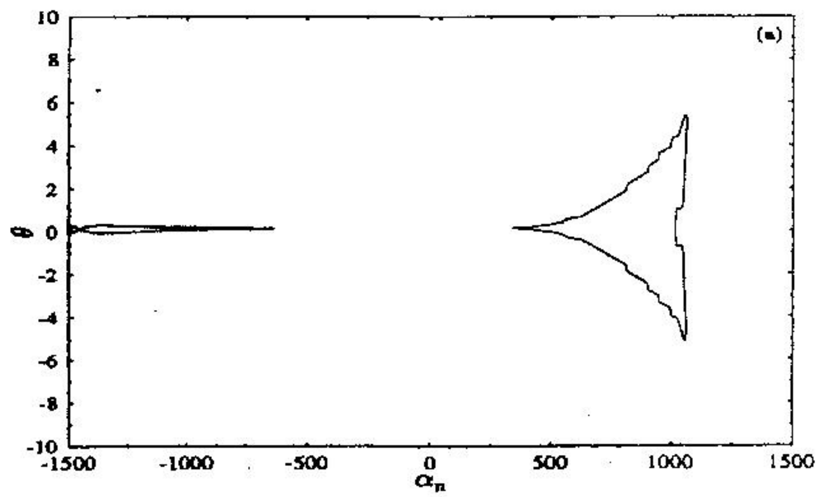

Figure 3: α_n versus θ for the case of $N = 0.1$ and (a)$\phi_m = 0$, (b)$\phi_m = \pi/2$ and (c)$\phi_m = \pi$

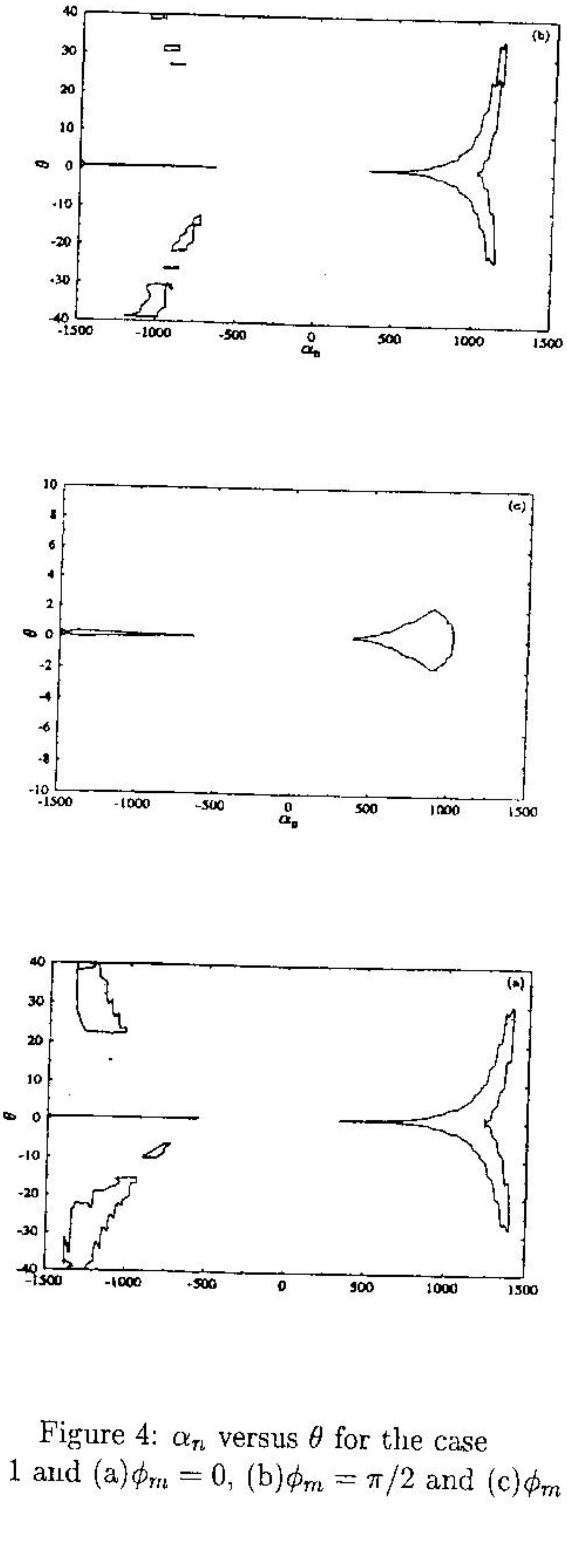

Figure 4: α_n versus θ for the case of $N = 1$ and (a)$\phi_m = 0$, (b)$\phi_m = \pi/2$ and (c)$\phi_m = \pi$

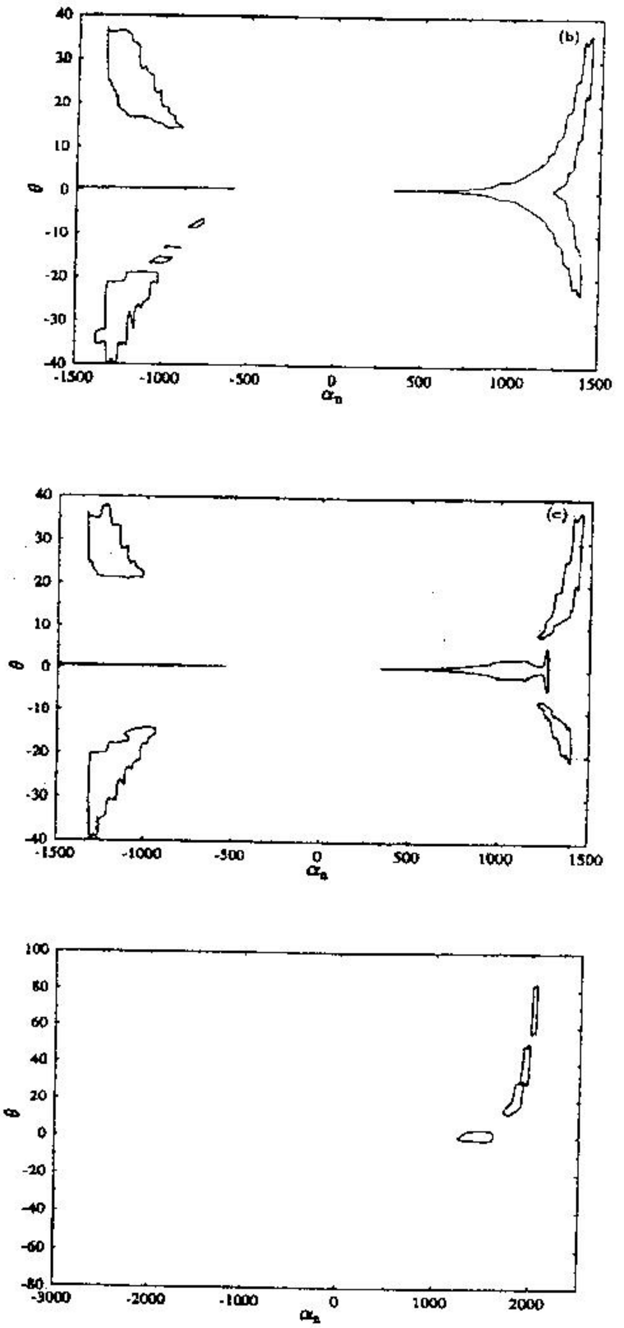

Figure 5: α_n versus θ for the case of $N = 5$ and$\phi_m = \pi$