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Free and Forced Vibrations of a Restrained Cantilever Beam Carrying a Concentrated Mass

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ABSTRACT The free and forced vibrations of a uniform cantilever beam with a translational elastic constraint at the beam tip and carrying a concentrated mass at an arbitrary intermediate point are analyzed. In the analysis, the base beam equation of motion is solved to obtain mode shape functions which satisfy all the geometric and natural boundary conditions at the beam ends. These functions are used in conjunction with Galerkin's method to obtain the free and the forced response. The key parameters are stiffness ratio, mass ratio and the position of the intermediate load. Partial computational results are compared with existing data: the agreement is good. For convenient use, the results are presented in dimensionless forms.

1. Introduction

The bending linear vibration of an elastically restrained beam element carrying concentrated masses located within the beam span is a subject of practical engineering interest and has been the objective of many recent theoretical investigations. Closed form solutions for this type of systems are generally difficult to obtain, and a number of researchers have considered various approximate methods for a variety of situations. Liu *et al.*[1] used Laplace transformation method to calculate the eigenvalues and eigenfunctions for a beam hinged at both ends by rotational springs and carrying arbitrary located concentrated masses. Liu and Haung[2] used the Laplace transformation method to study the free vibration of a beam hinged by a rotational spring at one end and carrying a concentrated mass at the tip, and another at an intermediate point. Ercoli and Laura[3] used Jacquot's method[4], Ritz method, and Rayleigh-Schmidt approach[5] to study the effect of an elastically mounted concentrated mass on the fundamental mode of a beam for various end conditions. Liu and Yeh[6] used Rayleigh-Ritz method in conjunction with beam functions satisfying all end conditions to study the free vibration of a restrained non-uniform beam with intermediate masses. Goel[7] studied the free vibration of a cantilever beam carrying a concentrated mass at an arbitrary intermediate location, and Kounadis[8] studied the free and forced vibrations of a restrained cantilever beam with attached masses. We note here that in all the aforementioned investigations each concentrated element (a lumped mass or a spring) located within the beam span was treated as a concentrated load on the beam. The mode shape functions obtained in these studies correspond to the 'base beam' (beam with prescribed boundary conditions but without intermediate concentrated elements). These approximate mode shape functions, however, are expected to deviate significantly from the true eigenfunctions as the stiffness or the inertia of an intermediate attached element becomes large compared with that of the base beam. Furthermore, the use of these mode shape functions as normal coordinates in the study of the forced vibration of these beam systems does not necessarily lead to uncoupled equations of motion and to closed form solutions.

The exact mode shape functions for this type of problems which account for all of the systems characteristics (*i.e.*, eigenfunctions which satisfy all ends boundary conditions and account for the effects of the intermediate concentrated elements) may be derived by dividing the beam into two segments at the point of attachment for each of the concentrated elements. One then formulates the equation of motion for each segment and requires the solution to each of these equations to satisfy all of the boundary and continuity conditions at the ends of the corresponding segment. This approach has been used by Gürgöze and Batan[9] and by Lau[10] to study the free vibration of a uniform cantilever beam with a rotational and translational constraint at some point. Kojima *et al.*[11] also used this approach to study the forced response of a cantilever beam carrying a tip mass and a magnetic vibration absorber at an intermediate location. Ebrahimi[12] used this approach to study the free and forced longitudinal vibrations of fixed-fixed bars with lumped masses. Although this approach yields the exact solutions and exact eigenfunctions for these types of problems, if suffers from the disadvantages of being algebraically cumbersome, *i.e.*, the application of this method to the solution of a beam problem with n intermediate concentrated elements involves the solution of n simultaneous boundary value problems and the solution of generally more complicated characteristic equation.

In the present work we study the free and forced vibrations of a cantilever beam constrained at the free end by a translational spring and carrying a concentrated mass at an arbitrary intermediate location, as shown in Fig. 1. The analysis involves solv-

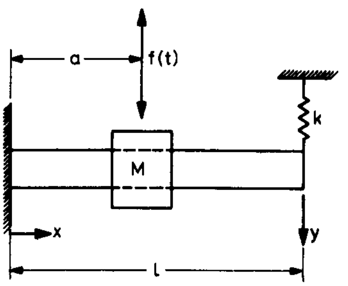

FIG. 1. Cantilever beam with an intermediate concentrated load and end translational spring.

ing the equation of motion for the elastically restrained base beam (no concentrated mass) to obtained the transcendental equation for the natural frequencies of the base beam and to obtain the mode shape functions which satisfy all of the base beam geometric and natural end conditions. These mode shape funcitons are then used in conjunction with Galerkin's method, treating the concentrated mass as an applied load, to study the free and forced vibration of the total beam system. Parametric studies are made for the effects of end spring stiffness, location and magnitude of the concentrated mass on the natural frequencies and resonance response of the system. The results of this approach are compared with the existing results of other methods. For convenience, all results are presented in dimensionless forms. All numerical computations were programmed using double precision on the VAX/VMS version 4.4 digital computer. Note that, although researchers have focused on a number of beam systems simialr to the one under consideration, no one, to the best of the author's knowledge, has adequately treated the present problem.

2. Analysis of the Problem

2.1 Equations of Motion

Adopting Bernoulli-Euler classical theory of bending of beams, the governing equation of motion for the uniform beam shown in Fig. 1 may be written as

$$EI\frac{\partial^4 y}{\partial x^4} + (m + M\delta(x-a))\frac{\partial^2 y}{\partial t^2} = \delta(x-a)f(t) \tag{1}$$

with the boundary conditions

$$\left.\begin{aligned} y(0) = \frac{\partial y(0)}{\partial x} &= 0 \\ \frac{\partial^2 y(l)}{\partial x^2} &= 0 \\ EI\frac{\partial^3 y(l)}{\partial x^3} &= ky(l) \end{aligned}\right\} \tag{2}$$

where E is the Young's modulus of elasticity, I is the second moment of the cross-sectional area, m is the mass per unit length of the beam, δ is the Dirac delta function, and $f(t)$ is an externally applied dynamic load. A series solution of equation (1) is assumed in the form

$$y(x,t) = \sum_{i=1}^{n} \phi_i(x)\, q_i(t) \tag{3}$$

where $\phi_i(x)$ are the mode shape funcitons (to be determined later) assumed here to satisfy all geometric and natural boundary conditions of the base beam, and $q_i(t)$ are generalized coordinates. Substituting equation (3) into equation (1) leads to

$$EI\sum_{i=1}^{n} \phi_i'''' q_i + m\sum_{i=1}^{n} \phi_i \ddot{q}_i = \delta(x-a)\left(f(t) - M\sum_{i=1}^{n} \phi_i \ddot{q}_i\right) \tag{4}$$

where a superscript prime and dot represent derivatives with respect to x and to time, respectively. Next multiplying equation (4) by ϕ_j, intergrating between the limits 0 and l (total length of beam) and using the boundary conditions given in equation (2), one obtains the Galerkin's equation[13] in martix form

$$[m]\{\ddot{q}(t)\} + [K]\{q(t)\} = \{Q\}f(t) \tag{5}$$

where $[m]$ and $[K]$ are $n \times n$ constant mass and stiffness matrices, respectively, and Q is $n \times 1$ generalized force matrix. The coefficients of these matrices are given by

$$\begin{aligned}
m_{ii} &= m \int_0^l \phi_i^2 dx + M\phi_i^2(a) && \text{(a)}\\
m_{ij} &= M\phi_i(a)\,\phi_j(a), \quad i \neq j && \text{(b)}\\
K_{ii} &= EI \int_0^l \phi_i''^2 dx + k\phi_i^2(l) && \text{(c)}\\
K_{ij} &= 0, \quad i \neq j && \text{(d)}\\
Q_i &= \phi_i(a) && \text{(e)}
\end{aligned} \tag{6}$$

Note that, the forms of $[K]$ and $[m]$ matrices in equation (5) depend on the mode shape functions $\phi_i(x)$ used to arrive at this equation. Since the $\phi_i(x)$ to be used in the present work are assumed to satisfy all natural and geometric boundary conditions at the beam ends, and since the beam has no elastic constraint at any intermediate location within its span then $K_{ij} = 0$ for $i \neq j$, *i.e.*, the $\phi_i(x)$ are orthogonal with respect to $[K]$ matrix. On the other hand, since these $\phi_i(x)$ do not include the effect of the concentrated mass M, the mass matrix $[m]$ in equation (5) is nondiagonal, *i.e.*, $\phi_i(x)$ are not orthogonal with respect to $[m]$ matrix since for $i \neq j$, $\int_0^l \phi_i \, \phi_j \, dx = 0$ and $M\phi_i(a)\,\phi_j(a) \neq 0$. The terms $M\phi_i(a)\,\phi_j(a)$ in the $[m]$ matrix are dynamic coupling terms and represent the effect of the concentrated mass M on the natural frequencies of the beam. If instead of the above mode shape funcitons $\phi_i(x)$ one calculates the $[m]$ and $[K]$ matrices in equation (5) using trial funcitons which satisfy some but not all of the beam end conditions, as for example when one uses Rayleigh-Ritz method, then both matrices $[m]$ and $[K]$ will in general be nondiagonal[13]. Before we proceed further and in order to check the accuracy of the formulation in equation (5), we calculate the eigenfunctions $\phi_i(x)$ for the base beam and solve the eigenvalue problem associated with equation (5) to determine the effects of the magnitude and location of the concentrated mass M, and the magnitude of the end spring stiffness on the natural frequencies of the beam-mass system.

2.2 Evaluation of Eigenfunctions ϕ_i and Natural Frequencies

First, the eigenfunctions $\phi_i(x)$ for the base beam are determined using the equation of motion

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = 0 \tag{7}$$

and the boundary conditions given in equation (2). Note that, the end spring k was not included in equation (7) but was accounted for as a natural boundary condition in equation (2). This was done in order to obtain the exact $\phi_i(x)$ which account for all the characteristics of the base beam. Using the standard method of separation of variables, one assumes

$$y(x, t) = Y(x) \cos \omega t \tag{8}$$

where ω is the eigenfrequency of the base beam. Substituting equation (8) into equation (7) leads to the following differential equation

$$Y''''(x) - q^4 Y(x) = 0 \tag{9}$$

where $q^4 = m\omega^2 / EI$. The solution to equation (9) is given by

$$Y(x) = C_1 \sin qx + C_2 \cos qx + C_3 \sinh qx + C_4 \cosh qx \tag{10}$$

Substituting the boundary conditions of equation (2) into equation (10), solving for the constants C_2, C_3 and C_4 in terms of C_1 and setting $\phi_i(x) = Y(x) / C_1$, one obtains the eigenfunctions

$$\phi_i = \sin qx - \sinh qx - r(\cos qx - \cosh qx) \tag{11}$$

and the transcendental equation for the natural frequencies

$$1 + \cos ql \cosh ql + \frac{S}{(ql)^3} (\cosh ql \sin ql - \sinh ql \cos ql) = 0 \tag{12}$$

where S is a dimensionless stiffness parameter defined as

$$S = \frac{kl^3}{EI} \tag{13}$$

which is a measure of the end spring stiffness relative to the base beam stiffness, and where

$$r = \frac{\sin ql + \sinh ql}{\cos ql + \cosh ql} \tag{14}$$

is the weighting constant associated with each mode. Note that for $S = 0$, *i.e.*, when $k = 0$, equation (12) reduces to that one obtained for a cantilever beam with free tip (see, Meirovitch[13] p. 227). Equation (12) has an infinite number of roots $q_i l$. To each root $q_i l$ corresponds a frequency ω_i and a mode shape $\phi_i(x)$. The first five frequency parameters $q_i l$ obtained by solving equation (12) numerically using the bisection method are compared in Table 1 with those obtained by Lau[10] for the cases S = 0,1,10,100,1000 and 10000. The agreement between the two results is exact, which is not surprising since the present method as well as the method of Lau[10] for the present case are exact.

3. Evaluation of [*M*], [*K*] and {*Q*} Matrices

The elements of the system matrices $[m]$, $[K]$ and $\{Q\}$ in equation (5) can now be evaluated since $q_i l$ and $\phi_i(x)$ for a given S needed for the evaluation of these matrices

TABLE 1. Frequency parameters $q_i l$ for the beam in Fig. 1 with $k \neq 0$ and $M = 0$.

	S					
	0	1	10	100	1000	10000
Present study	1.87510 4.69409 7.85476 10.99554 14.13717	2.01000 4.70379 7.85682 10.99629 14.13752	2.63893 4.79377 7.87565 11.00310 14.14072	3.64054 5.61600 8.08409 11.07484 14.17355	3.89780 6.87629 9.55253 11.95100 14.58153	3.92374 7.05070 10.15498 13.22183 16.22802
Reference [3]	1.87510 4.69409 7.86476 10.99554 14.13717	2.01000 4.70379 7.85682 10.99629 14.13752	2.63892 4.79377 7.87565 11.00310 14.14072	3.64054 5.61600 8.08409 11.07484 14.17355	3.89789 6.87629 9.55253 11.95100 14.58153	3.92374 7.05070 10.15498 13.22183 16.22802

are now known from the analysis of the previous section. Substituting equation (11) into equations (6-a) and (6-b), carrying out the integration and simplifying the results, one obtains

$$
\begin{aligned}
m_{ii} = m_b \{ r_i^2 + \frac{1}{2q_i l} [&(r_i^2 - 1) \sin q_i l \cos q_i l + (r_i^2 + 1) \sinh q_i l \cosh q_i l \\
&- 2(r_i^2 - 1) \cos q_i l \sinh q_i l - 2(r_i^2 + 1) \sin q_i l \cosh q_i l \\
&- 2r_i (\sin q_i l - \sinh q_i l)^2] \\
&+ \mu[\sin q_i l - \sinh q_i lz - r_i (\cos q_i lz - \cosh q_i lz)]^2 \}
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
m_{ij} = m_b \, \mu \{ &[\sin q_i lz - \sinh q_i lz - r_i (\cos q_i lz - \cosh q_i lz)] \\
&\times [\sin q_j lz - \sinh q_j lz - r_j (\cos q_j lz - \cosh q_j lz)] \}
\end{aligned} \tag{16}
$$

where m_b is the total mass of base beam, μ is a dimensionless mass parameter defined as $\mu = M/m_b$, and $z = a/l$, is a dimensionless parameter which defines the position a of the concentrated mass M relative to base beam total length l.

Similarly, substituting for equation (11) and its derivative into equation (6-c), integrating and simplifying the results, leads to

$$
\begin{aligned}
K_{ii} = \frac{EI}{l^3} \{ r_i^2 (q_i l)^4 + \frac{(q_i l)^3}{2} [&(r_i^2 - 1) \sin q_i l \cos q_i l + (r_i^2 + 1) \sinh q_i l \cosh q_i l \\
&+ 2(r_i^2 - 1) \cos q_i l \sinh q_i l + 2(1 + r_i^2) \sin q_i l \cosh q_i l - 2r_i (\sin q_i l + \sinh q_i l)^2] \\
&+ S [\sin q_i l - \sinh q_i l - r_i(\cos q_i l - \cosh q_i l)]^2 \}
\end{aligned} \tag{17}
$$

with $K_{ii} = 0$ for $i \neq j$. Finally, substituting equation (11) into equation (6-e), the generalized force coefficients Q_i becomes

$$
Q_i = \sin q_i lz - \sinh q_i lz - r_i (\cos q_i lz - \cosh q_i lz) \tag{18}
$$

Equaitons (15)-(18) are used in the next sections to evaluate the free and forced vibration of the beam-mass system for various values of the parameters S, μ, and z.

4. Effect of the Attached Mass M on Frequency Parameters $q_i l$

The combined effects of the system parameters S, μ, and z on the frequency parameters $q_i l$ for the beam-mass system can now be investigated from equation (5), since the system metrices $[m]$ and $[K]$ are now known from the analysis of the previous section. The frequency parameters $q_i l$ are obtained by solving the eigenvalue problem associated with equation (5), that is, one lets

$$[m]\{\ddot{q}(t)\} + [K]\{q(t)\} = 0 \tag{19}$$

A solution to the above linear equations may be obtained by assuming

$$\{q(t)\} = \{q\}\exp^{i\omega t} \tag{20}$$

where ω is now the eigenfrequency of the beam-mass system. Substituting equation (20) into (19) leads to the eigenvalue problem.

$$\left[[K] - \omega^2 [m] \right] \{q\} = 0 \tag{21}$$

The characteristic equation for the eigenfrequencies is obtained by setting the determinant of the coeffcients matrix in the above equation to zero,

$$\det \left[[K] - \omega^2 [m] \right] = 0 \tag{22}$$

For the sake of simplicity and the purpose of illustrating the procedure we evaluate only the first three $q_i l$; thus we carry out the expansion of the equation (22) for the case where $[m]$ and $[K]$ are of dimensions 3×3. For this case, the expansion of equation (22) leads, after simplifying, to the cubic characteristic equation

$$a_3X^3 + a_2X^2 + a_1X + a_0 = 0 \tag{23}$$

where X is a dimensionless parameter defined as

$$X = \frac{\omega^2 m_b l^3}{EI} = (q_i l)^4 \tag{24}$$

and a_3 , a_2 , a_1 and a_0 are dimensionless parameters expressed as follows

$$a_3 = \frac{(m_{11}\, m_{22}\, m_{33} - m_{11}\, m_{23}^2 + 2m_{12}\, m_{13}\, m_{23} - m_{12}^2\, m_{33} - m_{13}^2\, m_{22})}{m_b^3} \tag{25}$$

$$a_2 = (m_{23}^2K_{11} + m_{12}^2K_{33} + m_{13}^2K_{22} - m_{11}\, m_{22}K_{33} - m_{11}\, m_{33}K_{22} - m_{22}\, m_{33}K_{11}) \left(\frac{l^3}{EIm_b^2} \right) \tag{26}$$

$$a_1 = (m_{11}K_{22}K_{33} + m_{22}K_{11}K_{33} + m_{33}K_{11}K_{22}) \left(\frac{l^3}{EI} \right)^2 \left(\frac{1}{m_b} \right) \tag{27}$$

and

$$a_1 = -K_{11}K_{22}K_{33} \left(\frac{l^3}{EI} \right)^3 \tag{28}$$

Numerical solutions of equation (23) were found using the bisection method. It is interesting to note that it was found necessary to carry out all computations in equation (23) using double precision whereas when single precision computations were used the numerical solutions of equation (23) were found sometimes not to converge indicating the problem is very sensitive to small errors. Table 2 shows a comparison between the first three frequency parameters $q_i l$ obtained using equation (23) and those obtained in references[2,7,8], for the cases $\mu = 0.5$, 1, and 2;

TABLE 2. Frequency parameters $q_i l$ for the beam in Fig. 1 with $k = 0$ and $M \neq 0$.

μ	Present study	Reference [2]	Reference [7]	Reference [8]
$z = 0.3$				
0.5	1.857738 4.180531 6.922564	1.857729 4.174914 6.877625	No data	1.857687 4.174925 6.877645
$z = 0.5$				
0.5	1.778508 4.041218 7.854123	1.778434 4.032716 7.853989	1.792 4.106 7.838	No data
1.0	1.700566 3.784538 7.853956	1.700364 3.771661 7.853729	1.711 3.601 7.853	
2.0	1.581920 3.555561 7.853835	1.581490 3.539601 7.853520	1.597 3.503 7.837	
$z = 1.0$				
0.2	1.616519 4.273204 7.380543	1.616400 4.267060 7.318370	No data	No data
0.6	1.375923 4.096979 7.247670	1.375670 4.086650 7.172520		
1.0	1.248212 4.024738 7.215395	1.247920 4.031140 7.134113		
2.0	1.076506 3.995255 7.188919	1.076200 3.982570 7.102650		

$z = 0.3$, and 0.5, and $S = 0$, and for the cases $\mu = 0.2, 0.6$, and 2; $z = 1$, and $S = 0$. For the cases $z = 0.3$ and 0.5 the agreement is good while for the case $z = 1$ the agreement is fair. Note that, for $z = 1$ the concentrated mass M is located at the beam tip and an exact solution using the present method may be found for this case by simply replacing the third boundary condition $\frac{\partial^2 y(l)}{\partial x^2} = 0$ in equation (2) by the boundary condition $EI \frac{\partial^2 y(l)}{\partial x^2} = M \frac{\partial^2 y(l)}{\partial t^2} + ky(l)$. It is also be noted that the present results can be easily extended to the case where the beam may carry more than one intermediate concentrated element.

5. Forced Response

The forced response of the beam-mass system shown in Fig. 1 can be handled using the present procedure in a straightforward manner. Without loss of generality, we consider the case where $f(t)$ is a simple harmonic forcing function arising from a rotating mass unbalance, as this case is often encountered in many practical situations. Thus, we write $f(t)$ as

$$f(t) = m_0 e\Omega^2 \cos \Omega t \tag{29}$$

where m_0 is the mass of the rotating unbalance, e is the eccentricity of m_0 from the rotation axix, and Ω is the angular speed of rotation. Substituting equation (29) into equation (5) one obtains

$$[m]\{\ddot{q}\} + [K]\{q(t)\} = m_0 e\Omega^2 \{Q\} \cos \Omega t \tag{30}$$

where $[m]$, $[K]$ and $\{Q\}$ are known constant matrices given by equations (15) through (18). The solutions of the nonhomogeneous linear differential equations (30) may be obtained using any of the well known standard methods of linear algebra. One may wish to apply modal analysis to equation (30) and obtain the uncoupled forced equations of motion[13]. If the modal damping ratios are known, then one can easily introduce the damping terms in the uncoupled equations and carry out the damped response analysis. For the sake of simplicity, we solve equation (30) for the case where the dimension of $[m]$ and $[K]$ is 2×2. Using the impedance method[13] to solve equation (30), and using equations (15) through (18), one obtains the dimensionless equations

$$\bar{q}_1 = \frac{m_b q_1}{m_0 e} = \frac{\beta^2}{d_1}\left[\frac{(1 - b_1\beta^2)\,\phi_1(z) - b_2\beta^2\,\phi_2(z)}{(1 - b_1\beta^2)(1 - b_3\beta^2) - b_4\beta^4}\right] \tag{31}$$

$$\bar{q}_2 = \frac{m_b q_2}{m_0 e} = \frac{\beta^2}{d_2}\left[\frac{(1 - b_3\beta^2)\,\phi_2(z) - b_5\beta^2\,\phi_1(z)}{(1 - b_1\beta^2)(1 - b_3\beta^2) - b_4\beta^4}\right] \tag{32}$$

where $\bar{q}_1$ and $\bar{q}_2$ are dimensionless amplitudes, and β is a dimensionless frequency parameter defined as $\beta^2 = \dfrac{\Omega^2 m_b l^3}{EI}$, and the other parameters are also dimensionless defined as follows

$$\left.\begin{aligned}
b_1 &= \frac{m_{11}}{K_{22}}\left(\frac{EI}{m_b l^3}\right) & b_2 &= \frac{m_{12}}{K_{22}}\left(\frac{EI}{m_b l^3}\right) \\
b_3 &= \frac{m_{11}}{K_{11}}\left(\frac{EI}{m_b l^3}\right) & b_4 &= \frac{m_{12}^2}{K_{11}K_{22}}\left(\frac{EI}{m_b l^3}\right)^2 \\
b_5 &= \frac{m_{12}}{K_{11}}\left(\frac{EI}{m_b l^3}\right) & d_1 &= \frac{K_{11} l^3}{EI} \\
d_2 &= \frac{K_{22} l^3}{EI}
\end{aligned}\right\} \tag{33}$$

The total forced response dimensionless amplitude $\bar{y}(x, t)$ at a point (x) on the beam span is then, from equation (3), given by

$$\bar{y}(x) = \phi_1(x)\bar{q}_1 + \phi_2(x)\bar{q}_2 \tag{34}$$

where $\bar{q}_1$ and $\bar{q}_2$ are given by equations (31) and (32), respectively, and $\phi_1(x)$ and $\phi_2(x)$ are evaluated from equation (11) for $q_1 l$ and $q_2 l$, respectively. Note that, resonance occurs when the denominator in equations (31) and (32) is zero. Also note that, the parameter b_4 in the denominator of equation (31) and (32) represents the effect of the concentrated mass M on the resonance frequencies. If $m_{12} = 0$ then $b_4 = 0$ and the system equations of motion given by equation (30) are dynamically uncoupled so that resonance response occurs at $\beta^2 = 1/b_1$ and $\beta^2 = 1/b_3$. Since the b_4 β^4 term in the denominator of equations (31) and (32) is subtracted from the products of the first two terms, it represents a shift to the left (a decrease) in the resonance frequency, which is in agreement with the known physical fact that adding an inertia to a vibrating system leads to a decrease in system natural frequencies.

Figures 2 through 4 shows the variation of the dimensionless response amplitude $\bar{y}(x)$ at $x = l$ with the dimensionless frequency parameter β for various values of stiffness parameter S, mass ratio μ and relative position z of the concentrated mass and the rotating unbalance. All numerical computations were programmed on the VAX/VMS version 4.4 digital computer using double precision. It can be easily seen from these figures that increasing the stiffness ratio S results in an increase in the resonance frequecies, while increasing either mass ratio μ or the relative position z results in a decrease in the resonance frequencies, as one would expect. Note that, the above linear analysis is valid provided that the mass ratio μ is small enough not to produce large static deflection of the beam, as initial experimental work currently being carried out shows that the system in such a case may exhibits nonlinear and chaotic behavior.

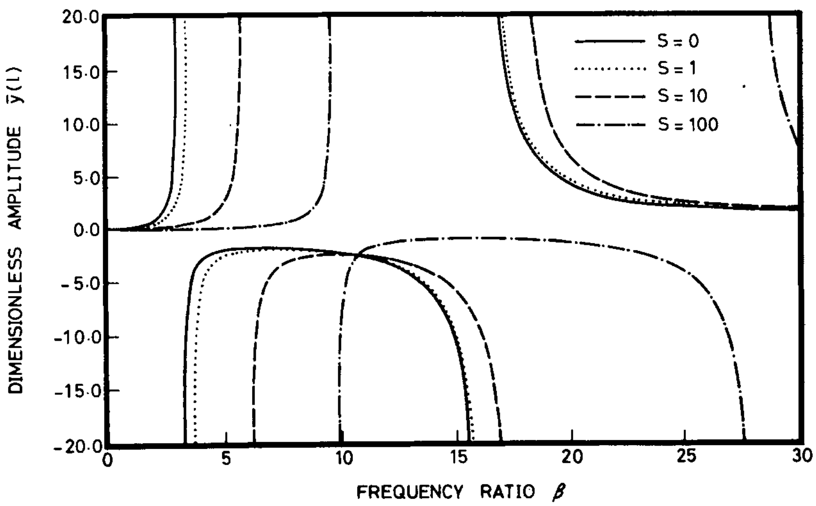

FIG. 2. Effect of variations of stiffness parameter S on the resonance response for mass ratio $\mu = 0.5$ and mass position $Z = 0.5$.

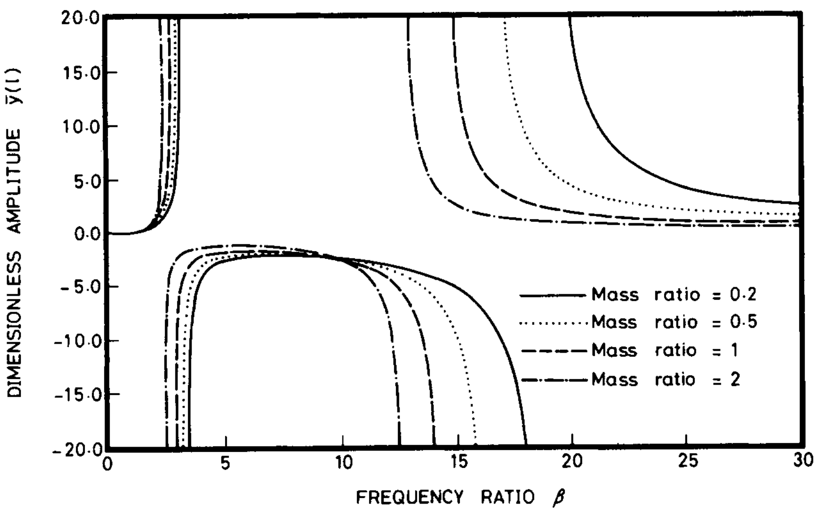

FIG. 3. Effect of variations of mass ratio μ on the resonance response for stiffness parameter $S = 0$ and mass position $Z = 0.5$.

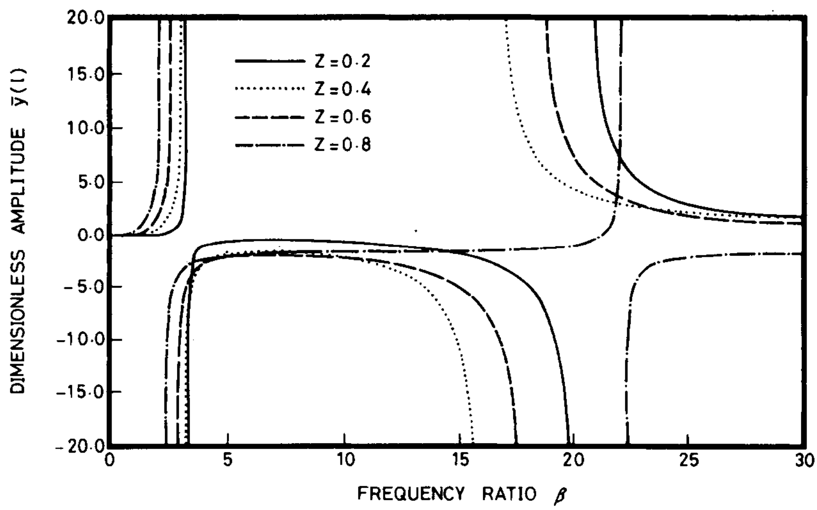

FIG. 4. Effect of variations of relative posiiton Z on the resonance response for stiffness $S = 0$ and mass ratio $\mu = 0.5$.

6. Conclusion

The effects of the stiffness of translational spring attached to the tip of a cantilever beam, and the effects of the magnitude and location of the intermediate concentrated mass on the free and forced vibrations of the beam are investigated. The analysis was carried out by treating the concentrated mass as an external loading and using Galerkin's method in conjunction with the mode shape functions which satisfy all natural and geometric boundary conditions of the beam. The results of the present method compared well with those of other methods, however, the present method has advantages in terms of computational effort, clarity and applicability to more complex systems. Such as beams with more complex boundary conditions and carrying elements having rotary inertia.

References

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الاهتزازات الحرة والقسرية لعتبة كابولية مقيدة الحركة وتحمل كتلة مركزة

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المستخلص . يقـدم هذا البحث دراسة نظرية للاهتزازات الحرة والقسرية لعتبة كابولية منتظمة ومقيدة عند طرفها الحر بزمبرك ذي حركة انتقالية ، وتحمل كتلة مركزة عند نقطة اختيارية . يتم في هذه الـدراسـة حل لمعـادلة العارضة الأساسية والحصول على دالات الشكل التي تحقق جميع الظروف الحدية الهندسية والطبيعية . وتستعمل هذه الدالات في طريقة Galerkin لدراسة الاهتزازات الحرة والقسرية . المعاملات الأساسية في هذا البحث هي نسبة الجسوءة ، نسبة الكتلة وموقع الحمل المتوسط . يتم مقارنة نتائج هذا البحث مع النتائج المنشورة في أبحاث أخرى مشابهة ، وتُقدَّم جميع نتائج هذا البحث بشكل لا بعدي .