



## Closure properties of operators on the Ma–Minda type starlike and convex functions <sup>☆</sup>

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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### ABSTRACT

A normalized univalent function  $f$  is called Ma–Minda starlike or convex if  $zf'(z)/f(z) \prec \varphi(z)$  or  $1 + zf''(z)/f'(z) \prec \varphi(z)$  where  $\varphi$  is a convex univalent function with  $\varphi(0) = 1$ . The class of Ma–Minda convex functions is shown to be closed under certain operators that are generalizations of previously studied operators. Analogous inclusion results are also obtained for subclasses of starlike and close-to-convex functions. Connections with various earlier works are made.

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### 1. Introduction and motivation

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disc in the complex plane and let  $\mathcal{A}$  denote the class of all functions  $f$  analytic in  $\mathbb{D}$  normalized by the conditions  $f(0) = 0$  and  $f'(0) = 1$ . An analytic function  $f$  is *subordinate* to  $g$  in  $\mathbb{D}$ , written  $f(z) \prec g(z)$  ( $z \in \mathbb{D}$ ), if there exists a function  $w$  analytic in  $\mathbb{D}$  with  $w(0) = 0$  and  $|w(z)| < 1$  satisfying  $f(z) = g(w(z))$ . In particular, if the function  $g$  is univalent in  $\mathbb{D}$ , then  $f(z) \prec g(z)$  is equivalent to  $f(0) = g(0)$  and  $f(\mathbb{D}) \subset g(\mathbb{D})$ . A function  $f \in \mathcal{A}$  is starlike if  $f(\mathbb{D})$  is a starlike domain with respect to 0, and  $f \in \mathcal{A}$  is convex if  $f(\mathbb{D})$  is a convex domain. Analytically, these geometric properties are respectively equivalent to the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{or} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

In terms of subordination, these conditions are respectively equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}.$$

The subclasses of  $\mathcal{A}$  consisting of starlike and convex functions are denoted respectively by  $\mathcal{ST}$  and  $\mathcal{CV}$ .

Ma and Minda [18] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function  $(1+z)/(1-z)$  by a more general analytic function  $\varphi$  with positive real part and normalized by the conditions  $\varphi(0) = 1$  and  $\varphi'(0) > 0$ . Further it is assumed that  $\varphi$  maps the unit disk  $\mathbb{D}$  onto a region starlike with respect

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